

DIFFERENTIAL-DIFFERENCE OPERATORS AND RADIAL PART FORMULAS FOR NON-INVARIANT ELEMENTS

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ABSTRACT. The classical radial part formula for the invariant differential operators and the K -invariant functions on a Riemannian symmetric space G/K is generalized to some non-invariant cases by use of Cherednik operators and a graded Hecke algebra \mathbf{H} naturally attached to G/K . We introduce a category \mathcal{C}_{rad} whose object is a pair of a $((\text{Lie } G)_{\mathbb{C}}, K)$ -module and an \mathbf{H} -module satisfying some axioms which are formally the same as the generalized Chevalley restriction theorem and the generalized radial part formula. Various pairs of analogous notions in the representation theories for G and \mathbf{H} , such as the Helgason-Fourier transform and the Opdam-Cherednik transform, are unified in terms of \mathcal{C}_{rad} . We construct natural functors which send an \mathbf{H} -module to a $((\text{Lie } G)_{\mathbb{C}}, K)$ -module and have some universal properties intimately related to \mathcal{C}_{rad} .

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1. INTRODUCTION

Let G be a real Lie group with a semisimple Lie algebra \mathfrak{g} and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ a fixed Iwasawa decomposition of \mathfrak{g} . We assume that the adjoint action $\text{Ad}(g)$ by any element $g \in G$ is an inner automorphism of the complexification $\mathfrak{g}_{\mathbb{C}}$ of \mathfrak{g} and that the closed subgroup $K := N_G(\mathfrak{k}) = \{g \in G; \text{Ad}(g)(\mathfrak{k}) \subset \mathfrak{k}\}$ is compact. (These assumptions are automatically satisfied if G is connected and its center is finite.) The *Harish-Chandra homomorphism* γ is the map of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ into the symmetric algebra $S(\mathfrak{a}_{\mathbb{C}})$ of $\mathfrak{a}_{\mathbb{C}}$ defined by

$$(1.1) \quad \begin{aligned} \gamma : U(\mathfrak{g}_{\mathbb{C}}) &= (\mathfrak{n}_{\mathbb{C}}U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}) \oplus U(\mathfrak{a}_{\mathbb{C}}) \\ &\xrightarrow{\text{projection}} U(\mathfrak{a}_{\mathbb{C}}) = S(\mathfrak{a}_{\mathbb{C}}) \xrightarrow{\text{shift by } -\rho} S(\mathfrak{a}_{\mathbb{C}}), \end{aligned}$$

where $S(\mathfrak{a}_{\mathbb{C}})$ is identified with the algebra of holomorphic polynomial functions on the dual space $\mathfrak{a}_{\mathbb{C}}^*$ of $\mathfrak{a}_{\mathbb{C}}$ and $\rho := \frac{1}{2} \text{Trace}(\text{ad}_{\mathfrak{n}}|_{\mathfrak{a}}) \in \mathfrak{a}_{\mathbb{C}}^*$. We have two different G -action $\ell(\cdot)$, $r(\cdot)$ on $C^\infty(G)$ defined by

$$\ell(g)f(x) = f(g^{-1}x), \quad r(g)f(x) = f(xg)$$

for $f \in C^\infty(G)$ and $g \in G$. Their differential actions are denoted by the same symbols. Thus, on the $\ell(G)$ -module

$$C^\infty(G/K) \simeq \{f(x) \in C^\infty(G); f(xk) = f(x) \text{ for } k \in K\},$$

$U(\mathfrak{g}_{\mathbb{C}})^K$ (the subalgebra of $\text{Ad}(K)$ -invariants in $U(\mathfrak{g}_{\mathbb{C}})$) naturally acts by $r(\cdot)$.

Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. A unique function $\phi_\lambda \in C^\infty(G/K)$ satisfying

$$(1.2) \quad \begin{cases} r(\Delta)\phi_\lambda = \gamma(\Delta)(\lambda)\phi_\lambda & \text{for } \Delta \in U(\mathfrak{g}_{\mathbb{C}})^K, \\ \ell(K)\text{-invariant}, \\ \phi_\lambda(1K) = 1 \end{cases}$$

is called a *spherical function*. Let A and N be respectively the analytic subgroups of \mathfrak{a} and \mathfrak{n} . Then from the global Iwasawa decomposition $G = NAK$ we have $G/K \simeq NA$. Let

$$\gamma_0 : C^\infty(G/K) \longrightarrow C^\infty(A)$$

be the natural restriction map and W the Weyl group for $(\mathfrak{g}, \mathfrak{a})$. Then the second property in (1.2) implies $\gamma_0(\phi_\lambda) \in C^\infty(A)^W$. By [HO], Heckman and Opdam started their studies on the *systems of hypergeometric differential equations*, which are certain modification of the system of differential equations satisfied by $\gamma_0(\phi_\lambda)$. At the early stage the existence of such modification had been quite non-trivial. But after a while *Cherednik operators* introduced by [Ch1] turned out to provide an elegant method to construct the modified systems. (This idea is due to [Hec2], in which *Heckman operators* play the same role as Cherednik operators.) In this context, a key fact is that the Cherednik operator $\mathcal{T} : S(\mathfrak{a}_{\mathbb{C}}) \rightarrow \text{End}_{\mathbb{C}} C^\infty(A)$ with a special parameter (Definition 4.2) satisfies for $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K$ and $f \in C^\infty(G/K)^{\ell(K)}$ a *radial part formula*

$$(1.3) \quad \gamma_0(r(\Delta)f) = \mathcal{T}(\gamma(\Delta))\gamma_0(f),$$

or equivalently,

$$(1.4) \quad \gamma_0(\ell(\Delta)f) = \mathcal{T}(\gamma(\theta\Delta))\gamma_0(f).$$

Here θ is the Cartan involution of G leaving K invariant. (The equivalence easily follows from the equality $f(\theta g^{-1}) = f(g)$ for $f \in C^\infty(G/K)^{\ell(K)}$.) In the first part of this paper (§§2, 4–7) we try to generalize (1.3) and (1.4) for non- K -invariant Δ and f .

First, we consider (1.3) concerns the radial part of the action of $r(U(\mathfrak{g}_\mathbb{C})^K)$ on $C^\infty(G/K)$ and generalize it to the case where $f \in C^\infty(G/K)$ is no longer K -invariant (§5). For example, (1.3) holds for any $\Delta \in U(\mathfrak{g}_\mathbb{C})^K$ and any K -finite $f \in C^\infty(G/K)$ such that all K -types in $\ell(U(\mathfrak{k}_\mathbb{C}))f$ are *single-petaled* (Theorem 5.1). A single-petaled K -type is a special kind of K -type introduced by [O]. We denote the set of single-petaled K -types by \widehat{K}_{sp} (Definition 2.1). In [Op1] Opdam studies the following system of differential-difference equations:

$$(1.5) \quad \mathcal{T}(\Delta)\varphi = \Delta(\lambda)\varphi \quad \forall \Delta \in S(\mathfrak{a}_\mathbb{C})^W.$$

Here $\lambda \in \mathfrak{a}_\mathbb{C}^*$ is any fixed spectral parameter and the unknown function $\varphi \in C^\infty(A)$ is not necessarily W -invariant. The *graded Hecke algebra* \mathbf{H} attached to the Iwasawa decomposition of $G = NAK$ (Definition 4.1) contains $S(\mathfrak{a}_\mathbb{C})$ and the group algebra $\mathbb{C}W$ of W as subalgebras and \mathcal{T} naturally extends to an algebra homomorphism $\mathbf{H} \rightarrow \text{End}_\mathbb{C} C^\infty(A)$. With respect to this \mathbf{H} -module structure the solution space $\mathcal{A}(A, \lambda)$ of (1.5) is generically an irreducible submodule of $C^\infty(A)$. The harmonic analysis developed by Opdam decomposes $C_c^\infty(A)$ (the space of compactly supported C^∞ functions on A) into the direct integral of $\mathcal{A}(A, \lambda)$'s (Proposition 14.15). His theory is surprisingly similar to the theory of the *Helgason-Fourier transform* for $C_c^\infty(G/K)$ (Proposition 14.9) where the G -module

$$\mathcal{A}(G/K, \lambda) := \{f \in C^\infty(G/K); r(\Delta)f = \gamma(\Delta)(\lambda)f \text{ for } \Delta \in U(\mathfrak{g}_\mathbb{C})^K\}$$

has a role of constitutional unit of $C^\infty(G/K)$. This G -module is known as the solution space for a maximal system of invariant differential operators on G/K (cf. [Hel4]). As a direct link between $\mathcal{A}(A, \lambda)$ and $\mathcal{A}(G/K, \lambda)$, we have a linear bijection:

$$(1.6) \quad \gamma_0 : \mathcal{A}(G/K, \lambda)^{\ell(K)} \xrightarrow{\sim} \mathcal{A}(A, \lambda)^W; \quad \phi_\lambda \longmapsto \gamma_0(\phi_\lambda).$$

This comes from the Chevalley restriction theorem, Harish-Chandra's celebrated exact sequence

$$(1.7) \quad 0 \rightarrow (U(\mathfrak{g}_\mathbb{C})\mathfrak{k}_\mathbb{C})^K \rightarrow U(\mathfrak{g}_\mathbb{C})^K \xrightarrow{\gamma} S(\mathfrak{a}_\mathbb{C})^W \rightarrow 0,$$

and (1.3). Now a generalization of Chevalley restriction theorem given in [O] asserts that γ_0 naturally induces for each $V \in \widehat{K}_{\text{sp}}$ a linear bijection

$$(1.8) \quad \Gamma_0^V : \text{Hom}_K(V, C^\infty(G/K)) \xrightarrow{\sim} \text{Hom}_W(V^M, C^\infty(A)).$$

Here M is the centralizer of A in K and V^M is the M -fixed part of V (see §2). This, combined with (1.7) and our generalization of (1.3), produces a link stronger than (1.6).

That is, for each $V \in \widehat{K}_{\text{sp}}$ it holds that

$$(1.9) \quad \Gamma_0^V : \text{Hom}_K(V, \mathcal{A}(G/K, \lambda)) \xrightarrow{\sim} \text{Hom}_W(V^M, \mathcal{A}(A, \lambda))$$

(Corollary 5.2). This means the generalized Chevalley restriction theorem respects the spectra.

Secondly, we consider (1.4) concerns the radial part of the action of $\ell(U(\mathfrak{g}_{\mathbb{C}}))$ on $C^\infty(G/K)$ and generalize it to the case where neither operators nor functions are assumed K -invariant (§7). By use of a certain involutive automorphism $\theta_{\mathbf{H}}$ of \mathbf{H} (Definition 7.1), we can rewrite (1.4) as

$$\gamma_0(\ell(\Delta)f) = \mathcal{T}(\theta_{\mathbf{H}}\gamma(\Delta))\gamma_0(f).$$

So it is natural to think of the \mathbf{H} -action $\mathcal{T}(\theta_{\mathbf{H}}\cdot)$ on $C^\infty(A)$ as the radial counterpart of the G -action $\ell(\cdot)$ on $C^\infty(G/K)$. With respect to these module structures of $C^\infty(G/K)$ and $C^\infty(A)$ let us apply the Frobenius reciprocity to both the sides of (1.8). We then get a linear bijection

$$\Gamma_0 : \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V, C^\infty(G/K)_{K\text{-finite}}) \xrightarrow{\sim} \text{Hom}_{\mathbf{H}}(\mathbf{H} \otimes_{\mathbb{C}W} V^M, C^\infty(A)).$$

Here the subscript “ K -finite” indicates the subspace consisting of K -finite vectors. To formulate our generalization, we need also to generalize the Harish-Chandra homomorphism γ . Suppose E is another single-petaled K -type. Then we can define a natural map

$$\Gamma_V^E : \text{Hom}_K(E, U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V) \rightarrow \text{Hom}_W(E^M, \mathbf{H} \otimes_{\mathbb{C}W} V^M)$$

(§6). The case where V is the trivial K -type \mathbb{C}_{triv} is studied in [O] and the case where $E = V = \mathbb{C}_{\text{triv}}$ reduces to γ . Let us now state our generalization of (1.4). For any $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V, C^\infty(G/K)_{K\text{-finite}}) \simeq \text{Hom}_K(V, C^\infty(G/K))$ and any $\Psi \in \text{Hom}_K(E, U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)$ it holds that

$$(1.10) \quad \Gamma_0^E(\Phi \circ \Psi) = \Gamma_0(\Phi) \circ \Gamma_V^E(\Psi).$$

In other words, if we take a basis $\{v_1, \dots, v_n\}$ of V so that $\{v_1, \dots, v_m\}$ is a basis of V^M ($m \leq n$) and if we write for any $e \in E^M$

$$\Psi[e] = \sum_{i=1}^n D_i \otimes v_i \text{ with } D_i \in U(\mathfrak{g}_{\mathbb{C}}), \quad \Gamma_V^E(\Psi)[e] = \sum_{i=1}^m h_i \otimes v_i \text{ with } h_i \in \mathbf{H},$$

then

$$\gamma_0\left(\sum_{i=1}^n \ell(D_i)\Phi[v_i]\right) = \sum_{i=1}^m \mathcal{T}(\theta_{\mathbf{H}}h_i)\gamma_0(\Phi[v_i]).$$

Actually we have a further generalization of (1.10) to the case where E and V are *quasi-single-petaled* K -types (Definition 2.1). The complete results will be stated in Theorem 7.3. For simplicity, in this introductory section we shall state all results without using the notion of quasi-single-petaled K -types.

In section 8 we define a natural correspondence

$$(1.11) \quad \Xi_0^{\min} : \{\mathbf{H}\text{-submodule of } C^\infty(A)\} \rightarrow \{(\mathfrak{g}_{\mathbb{C}}, K)\text{-submodule of } C^\infty(G/K)_{K\text{-finite}}\}$$

using (1.8). For example, Ξ_0^{\min} maps a unique irreducible \mathbf{H} -submodule $X_{\mathbf{H}}(\lambda)$ of $\mathcal{A}(A, \lambda)$ to a unique irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule $X_G(\lambda)$ of $\mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ (Theorem 11.6). (The module structures of $\mathcal{A}(A, \lambda)$ and $\mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ will be studied in §11.) If \mathcal{X} is an \mathbf{H} -submodule of $C^\infty(A)$ then for each $V \in \widehat{K}_{\text{sp}}$ the linear map Γ_0^V induces a linear bijection

$$\Gamma_0^V : \text{Hom}_K(V, \Xi_0^{\min}(\mathcal{X})) \xrightarrow{\sim} \text{Hom}_W(V^M, \mathcal{X})$$

(Theorem 8.23 (iii)). This property comes from (1.10). Now we can develop a similar story for the pair $(U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathbb{C}_{\text{triv}}, \mathbf{H} \otimes_{\mathbb{C}W} \mathbb{C}_{\text{triv}})$ instead of $(C^\infty(G/K)_{K\text{-finite}}, C^\infty(A))$. Namely, we can define a correspondence

(1.12)

$$\Xi^{\min} : \{\mathbf{H}\text{-submodule of } \mathbf{H} \otimes_{\mathbb{C}W} \mathbb{C}_{\text{triv}}\} \rightarrow \{(\mathfrak{g}_{\mathbb{C}}, K)\text{-submodule of } U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathbb{C}_{\text{triv}}\}$$

for which $\Gamma_{\mathbb{C}_{\text{triv}}}^V$ has the same property with Γ_0^V . Motivated by this parallelism, we introduce a new category \mathcal{C}_{rad} (Definitions 8.1, 8.3 and 8.5). An object $\mathcal{M} \in \mathcal{C}_{\text{rad}}$, which we call a *radial pair*, is a pair of a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module \mathcal{M}_G and an \mathbf{H} -module $\mathcal{M}_{\mathbf{H}}$ satisfying a set of axioms which are formally the same as the generalized Chevalley restriction theorem and the second type of radial part formula. Some parts of the axioms are as follows: To each $V \in \widehat{K}_{\text{sp}}$ there attach a linear map

$$\tilde{\Gamma}_{\mathcal{M}}^V : \text{Hom}_K(V, \mathcal{M}_G) \rightarrow \text{Hom}_W(V^M, \mathcal{M}_{\mathbf{H}})$$

and a subspace $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$ of $\text{Hom}_K(V, \mathcal{M}_G)$ such that the restriction of $\tilde{\Gamma}_{\mathcal{M}}^V$ to $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$ gives a bijection

$$\tilde{\Gamma}_{\mathcal{M}}^V : \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \xrightarrow{\sim} \text{Hom}_W(V^M, \mathcal{M}_{\mathbf{H}})$$

(cf. (1.8)) and for any $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$, $E \in \widehat{K}_{\text{sp}}$ and $\Psi \in \text{Hom}_K(E, U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)$ it holds that

$$\tilde{\Gamma}_{\mathcal{M}}^E(\Phi \circ \Psi) = \tilde{\Gamma}_{\mathcal{M}}(\Phi) \circ \Gamma_V^E(\Psi)$$

(cf. (1.10)); Here $\tilde{\Gamma}_{\mathcal{M}}(\Phi)$ is a morphism in $\text{Hom}_{\mathbf{H}}(\mathbf{H} \otimes_{\mathbb{C}W} V^M, \mathcal{M}_{\mathbf{H}})$ identified with $\tilde{\Gamma}_{\mathcal{M}}^V(\Phi) \in \text{Hom}_W(V^M, \mathcal{M}_{\mathbf{H}})$ by the Frobenius reciprocity. In many important cases $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) = \text{Hom}_K(V, \mathcal{M}_G)$ (cf. Remark 8.8). For any $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{rad}}$ a correspondence

$$\Xi_{\mathcal{M}}^{\min} : \{\mathbf{H}\text{-submodule of } \mathcal{M}_{\mathbf{H}}\} \rightarrow \{(\mathfrak{g}_{\mathbb{C}}, K)\text{-submodule of } \mathcal{M}_G\}.$$

is defined. If \mathcal{X} is an \mathbf{H} -submodule of $\mathcal{M}_{\mathbf{H}}$ then $(\Xi_{\mathcal{M}}^{\min}(\mathcal{X}), \mathcal{X})$ is again a radial pair (Theorem 8.23 (ii)).

Besides $(C^\infty(G/K)_{K\text{-finite}}, C^\infty(A))$ and $(U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathbb{C}_{\text{triv}}, \mathbf{H} \otimes_{\mathbb{C}W} \mathbb{C}_{\text{triv}})$, there are many natural radial pairs. For example, (1.9) implies $(\mathcal{A}(G/K, \lambda)_{K\text{-finite}}, \mathcal{A}(A, \lambda)) \in \mathcal{C}_{\text{rad}}$ (Example 8.25). Also, $B(\lambda) = (B_G(\lambda)_{K\text{-finite}}, B_{\mathbf{H}}(\lambda)) \in \mathcal{C}_{\text{rad}}$ (Theorem 9.8) where $B_G(\lambda)$ and $B_{\mathbf{H}}$ are the minimal spherical principal series representations for G and \mathbf{H} induced from the same character $-\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. In §§9–14 we shall see various pairs of analogous notions appearing in the representation theories for G and \mathbf{H} can be peacefully packed in the category \mathcal{C}_{rad} . In §12 we define an \mathbf{H} -homomorphism $\mathcal{P}_{\mathbf{H}}^\lambda :$

$B_{\mathbf{H}}(\lambda) \rightarrow \mathcal{A}(A, \lambda)$ analogous to the *Poisson transform* $\mathcal{P}_G^\lambda : B_G(\lambda) \rightarrow \mathcal{A}(G/K, \lambda)$, and then prove they constitute a morphism in \mathcal{C}_{rad} :

$$\mathcal{P}^\lambda = (\mathcal{P}_{\mathbf{H}}^\lambda, \mathcal{P}_G^\lambda) : (B_G(\lambda)_{K\text{-finite}}, B_{\mathbf{H}}(\lambda)) \rightarrow (\mathcal{A}(G/K, \lambda)_{K\text{-finite}}, \mathcal{A}(A, \lambda))$$

(Theorem 12.3). As a morphism in \mathcal{C}_{rad} (cf. Definition 8.1), \mathcal{P}^λ satisfies for any $V \in \widehat{K}_{\text{sp}}$ and $\Phi \in \text{Hom}_K(V, B_G(\lambda))$

$$\Gamma_0^V(\mathcal{P}_G^\lambda \circ \Phi) = \mathcal{P}_{\mathbf{H}}^\lambda \circ \tilde{\Gamma}_{B(\lambda)}^V(\Phi).$$

In §13 we study a *Knapp-Stein type intertwining operator*

$$\tilde{\mathcal{A}}_G(w, \lambda) : B_G(\lambda) \rightarrow B_G(w\lambda) \quad (w \in W)$$

and its analogue $\tilde{\mathcal{A}}_{\mathbf{H}}(w, \lambda)$ in the category $\mathbf{H}\text{-Mod}$ of \mathbf{H} -modules. Of course they constitute a morphism in \mathcal{C}_{rad} (Theorem 13.5). In §14 we study the relation between the Helgason-Fourier transform \mathcal{F}_G and the *Opdam-Cherednik transform* $\mathcal{F}_{\mathbf{H}}$ introduced respectively in [Hel1] and [Op1]. The Paley-Wiener theorems, the inversion formulas and the Plancherel formulas for both transforms can be successfully combined in \mathcal{C}_{rad} (Theorem 14.16).

In §15 we prove the generalized Chevalley restriction theorem holds for the class \mathcal{A} of analytic functions (Theorem 15.4). This result implies that $(\mathcal{A}(G/K), \mathcal{A}(A)) \in \mathcal{C}_{\text{rad}}$ (Corollary 15.5) and that the correspondence (1.11) restricts to

$$(1.13) \quad \Xi_0^{\min} : \{\mathbf{H}\text{-submodule of } \mathcal{A}(A)\} \rightarrow \{(\mathfrak{g}_{\mathbb{C}}, K)\text{-submodule of } \mathcal{A}(G/K)_{K\text{-finite}}\}.$$

In §§16–18 we shall construct three functors Ξ_{rad} , Ξ^{\min} and Ξ , each of which sends an \mathbf{H} -module \mathcal{X} to a (\mathfrak{g}, K) -module \mathcal{Y} such that $(\mathcal{Y}, \mathcal{X}) \in \mathcal{C}_{\text{rad}}$. If $\mathcal{X} \in \mathbf{H}\text{-Mod}$ then there is a sequence of surjective (\mathfrak{g}, K) -homomorphisms

$$\Xi_{\text{rad}}(\mathcal{X}) \twoheadrightarrow \Xi^{\min}(\mathcal{X}) \twoheadrightarrow \Xi(\mathcal{X}).$$

These functors have their own universal properties. First, the functor $\mathbf{H}\text{-Mod} \ni \mathcal{X} \mapsto (\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$ is left adjoint to the functor $\mathcal{C}_{\text{rad}} \ni (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \mapsto \mathcal{M}_{\mathbf{H}} \in \mathbf{H}\text{-Mod}$ (Theorem 16.4). If $\mathcal{X} \in \mathbf{H}\text{-Mod}$ has a *central character* (Definition 16.5) then $\Xi_{\text{rad}}(\mathcal{X}) \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ has a corresponding *infinitesimal character* (Theorem 16.6). For a finite-dimensional $\mathcal{X} \in \mathbf{H}\text{-Mod}$ the length of $\Xi_{\text{rad}}(\mathcal{X})$ is finite (Theorem 16.7). Secondly, Ξ^{\min} is a functor extending the correspondence (1.12) (Corollary 17.4). If $\mathcal{X} \in \mathbf{H}\text{-Mod}$ has finite dimension, then $\Xi^{\min}(\mathcal{X})$ can be embedded into the G -module induced from \mathcal{X} viewed naturally as an MAN -module (Theorem 17.6). Using this realization we can prove that if $(\cdot, \cdot)^{\mathbf{H}}$ is an invariant sesquilinear form on two finite-dimensional \mathbf{H} -modules $\mathcal{X}_1, \mathcal{X}_2$ (cf. Definition 9.4) then there exists a natural invariant sesquilinear form $(\cdot, \cdot)^G$ on $\Xi^{\min}(\mathcal{X}_1) \times \Xi^{\min}(\mathcal{X}_2)$ (Theorem 17.8). Finally, Ξ extends the correspondence (1.13) (Corollary 18.5 (iii)). If $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{rad}}$ then there exists a unique $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I}_G : \Xi_{\mathcal{M}}^{\min}(\mathcal{M}_{\mathbf{H}}) \rightarrow \Xi(\mathcal{M}_{\mathbf{H}})$ such that

$$(\mathcal{I}_G, \text{id}_{\mathcal{M}_{\mathbf{H}}}) : (\Xi_{\mathcal{M}}^{\min}(\mathcal{M}_{\mathbf{H}}), \mathcal{M}_{\mathbf{H}}) \rightarrow (\Xi(\mathcal{M}_{\mathbf{H}}), \mathcal{M}_{\mathbf{H}})$$

is a morphism in \mathcal{C}_{rad} (Theorem 18.4 (ii)). If $(\cdot, \cdot)^{\mathbf{H}}$ is as above then $(\cdot, \cdot)^G$ induces a sesquilinear form on $\Xi(\mathcal{X}_1) \times \Xi(\mathcal{X}_2)$ (Theorem 18.6). This form is non-degenerate

when $(\cdot, \cdot)^{\mathbf{H}}$ is non-degenerate. In §19 we shall restrict ourselves to the case of $G = SL(2, \mathbb{R})$ and completely describe the behaviors of these functors for the irreducible \mathbf{H} -modules (Theorem 19.2). Our three functors are closely related to the functors given by [AS, EFM, CT]. Roughly speaking, our functors are right inverses of their functors. The author wants to discuss the relations between them in a subsequent paper.

Now, let us return to the first part of the paper and consider the infinitesimal (or “tangential”) version of radial part formulas. Let \mathfrak{s} be the -1 -eigenspace of θ in \mathfrak{g} . For the *Cartan motion group* $G_{\text{CM}} := K \ltimes \mathfrak{s}$ and the *rational Dunkl operators* introduced by [Dun], we have similar results to the case of G/K . Using the K -module isomorphism

$$(1.14) \quad C^\infty(G/K) \simeq C^\infty(\mathfrak{s}); \quad f \longmapsto f(\exp \cdot).$$

and the W -module isomorphism

$$(1.15) \quad C^\infty(A) \simeq C^\infty(\mathfrak{a}); \quad \varphi \longmapsto \varphi(\exp \cdot),$$

we identify γ_0 with the natural restriction map $C^\infty(\mathfrak{s}) \rightarrow C^\infty(\mathfrak{a})$. Via the Killing form $B(\cdot, \cdot)$ of \mathfrak{g} , $S(\mathfrak{s}_{\mathbb{C}})$ is identified with the algebra $\mathcal{P}(\mathfrak{s})$ of polynomial functions on \mathfrak{s} , and $S(\mathfrak{a}_{\mathbb{C}})$ with $\mathcal{P}(\mathfrak{a})$. Thus γ_0 induces a map $S(\mathfrak{s}_{\mathbb{C}}) \rightarrow S(\mathfrak{a}_{\mathbb{C}})$, which is also denoted by γ_0 . For $X \in \mathfrak{s}$ let $\partial(X)$ denote the X -directional derivative operator on \mathfrak{s} . Extend the linear map $\partial : \mathfrak{s} \rightarrow \text{End}_{\mathbb{C}} C^\infty(\mathfrak{s})$ to an algebra homomorphism $\partial : S(\mathfrak{s}_{\mathbb{C}}) \rightarrow \text{End}_{\mathbb{C}} C^\infty(\mathfrak{s})$. In [Je] de Jeu gives a simple proof for the fact that the Dunkl operator $\mathcal{D} : S(\mathfrak{a}_{\mathbb{C}}) \rightarrow \text{End}_{\mathbb{C}} C^\infty(\mathfrak{a})$ with a special parameter (Definition 3.1) satisfies a radial part formula

$$(1.16) \quad \gamma_0(\partial(\Delta)f) = \mathcal{D}(\gamma_0(\Delta))\gamma_0(f)$$

for $\Delta \in S(\mathfrak{s}_{\mathbb{C}})^K$ and $f \in C^\infty(\mathfrak{s})^K$. In §3 we show that (1.16) holds for more general combinations of Δ and f (Theorems 3.4, 3.5). But we do not try to for maximally possible combinations because in this paper our stress is persistently on the case of G/K . Nevertheless, after recalling the generalized Chevalley restriction theorem in §2, we shall discuss the case of G_{CM}/K first in §3. One reason for it is the easiness: de Jeu’s simple method still works in our generalization without any change. Another reason is that the resulting radial part formulas will be good prototypes for the Riemannian symmetric space case in subsequent sections.

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2. THE CHEVALLEY RESTRICTION THEOREM, I

In this section we review a generalization of the Chevalley restriction theorem given in [O]. Let Σ be the restricted root system for $(\mathfrak{g}, \mathfrak{a})$, and Σ^+ the positive system of Σ corresponding to \mathfrak{n} . The root space for each $\alpha \in \Sigma$ is denoted by \mathfrak{g}_α and we fix $E_\alpha \in \mathfrak{g}_\alpha$ so that $-B(E_\alpha, \theta E_\alpha) = \frac{2}{B(H_\alpha, H_\alpha)}$. Here for $\mu \in \mathfrak{a}_{\mathbb{C}}^*$, H_μ denotes a unique element in $\mathfrak{a}_{\mathbb{C}}$

such that $\mu(H) = B(H_\mu, H)$ for $H \in \mathfrak{a}$. Putting $Z_\alpha = E_\alpha + \theta E_\alpha \in \mathfrak{k}$, we introduce some classes of K -types.

Definition 2.1 (K -types). (i) In this paper by the terminology “ K -type” we mean an irreducible unitary representation of K or its equivalence class. The set of K -types is denoted by \widehat{K} .

(ii) $V \in \widehat{K}$ is called M -spherical if $V^M \neq \{0\}$. The set of M -spherical K -types is denoted by \widehat{K}_M .

(iii) For $V \in \widehat{K}_M$ put

$$(2.1) \quad V_{\text{single}}^M = \{v \in V^M; Z_\alpha(Z_\alpha^2 + 4)v = 0 \text{ for any } \alpha \in \Sigma\}.$$

$V \in \widehat{K}_M$ is called *quasi-single-petaled* if $V_{\text{single}}^M \neq \{0\}$. The set of quasi-single-petaled K -types is denoted by \widehat{K}_{qsp} .

(iv) $V \in \widehat{K}_M$ is called *single-petaled* if $V_{\text{single}}^M = V^M$. The set of single-petaled K -types is denoted by \widehat{K}_{sp} .

Remark 2.2. For $V \in \widehat{K}_M$, V^M is naturally a W -module. Its subspace V_{single}^M is W -stable since the definition (2.1) is independent of the choice of E_α 's (cf. [O, Remark 1.2]).

Let \mathcal{F} be either one of the following function classes on G/K or A : C^∞ (smooth functions), C_c^∞ (smooth functions with compact support), \mathcal{P} (polynomial functions on $\mathfrak{s} \simeq G/K$ or $\mathfrak{a} \simeq A$).

Theorem 2.3 (the generalized Chevalley restriction theorem [O]). *Suppose $V \in \widehat{K}_M$.*

(i) *The map*

$$(2.2) \quad \begin{aligned} \Gamma_0^V : \text{Hom}_K(V, \mathcal{F}(G/K)) &\longrightarrow \text{Hom}_W(V^M, \mathcal{F}(A)); \\ \Phi &\longmapsto (\varphi : V^M \hookrightarrow V \xrightarrow{\Phi} \mathcal{F}(G/K) \xrightarrow{\gamma_0} \mathcal{F}(A)) \end{aligned}$$

is well defined.

(ii) *Let $p^V : V \rightarrow V^M$ be the orthogonal projection with respect to a K -invariant inner product of V . Then for any $\Phi \in \text{Hom}_K(V, \mathcal{F}(G/K))$, $\gamma_0 \circ \Phi[v] = \Gamma_0^V(\Phi) \circ p^V[v]$.*

(iii) Γ_0^V *is injective.*

(iv) *Define*

$$V_{\text{double}}^M = V^M \cap \sum \{m Z_\alpha(Z_\alpha^2 + 4)V; \alpha \in \Sigma, m \in M\}.$$

Then $V^M = V_{\text{single}}^M \oplus V_{\text{double}}^M$ is a direct sum decomposition into two W -submodules. A W -submodule $U \subset V^M$ satisfies the condition

$$\text{Im } \Gamma_0^V \supset \{\varphi \in \text{Hom}_W(V^M, \mathcal{F}(A)); \varphi[U] = \{0\}\}$$

if and only if $U \supset V_{\text{double}}^M$. In particular $\text{Im } \Gamma_0^V$ contains any $\varphi \in \text{Hom}_W(V^M, \mathcal{F}(A))$ such that $\varphi[V_{\text{double}}^M] = \{0\}$.

(v) Γ_0^V *is surjective if and only if $V \in \widehat{K}_{\text{sp}}$.*

Remark 2.4. In view of (1.14) and (1.15), Γ_0^V is naturally identified with the following map:

$$\begin{aligned}\Gamma_0^V : \text{Hom}_K(V, \mathcal{F}(\mathfrak{s})) &\longrightarrow \text{Hom}_W(V^M, \mathcal{F}(\mathfrak{a})); \\ \Phi &\longmapsto (\varphi : V^M \hookrightarrow V \xrightarrow{\Phi} \mathcal{F}(\mathfrak{s}) \xrightarrow{\gamma_0} \mathcal{F}(\mathfrak{a})).\end{aligned}$$

In fact, [O] originally proved Theorem 2.3 in this form.

Definition 2.5. (i) For $V \in \widehat{K}_M$ we put

$$\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{F}(G/K)) = \{\Phi \in \text{Hom}_K(V, \mathcal{F}(G/K)); \Gamma_0^V(\Phi)[V_{\text{double}}^M] = \{0\}\}.$$

$\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{F}(\mathfrak{s})) \simeq \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{F}(G/K))$ is similarly defined. (This symbol comes from the analogy between $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{F}(G/K))$ and $\text{Hom}_K^{2 \rightarrow 2}(V, U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathbb{C}_{\text{triv}})$. The latter will be defined in §6.)

(ii) We define the map

$$\begin{aligned}\tilde{\Gamma}_0^V : \text{Hom}_K(V, \mathcal{F}(G/K)) &\longrightarrow \text{Hom}_W(V_{\text{single}}^M, \mathcal{F}(A)); \\ \Phi &\longmapsto (\varphi : V_{\text{single}}^M \hookrightarrow V \xrightarrow{\Phi} \mathcal{F}(G/K) \xrightarrow{\gamma_0} \mathcal{F}(A)).\end{aligned}$$

This is also identified with $\tilde{\Gamma}_0^V : \text{Hom}_K(V, \mathcal{F}(\mathfrak{s})) \longrightarrow \text{Hom}_W(V_{\text{single}}^M, \mathcal{F}(\mathfrak{a}))$.

Remark 2.6. (i) If $V \in \widehat{K}_{\text{sp}}$ then $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{F}(G/K)) = \text{Hom}_K(V, \mathcal{F}(G/K))$. If $V \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$ then $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{F}(G/K)) = \{0\}$.

(ii) If $V \in \widehat{K}_M$ then it follows from Theorem 2.3 that the restriction of $\tilde{\Gamma}_0^V$ to $\text{Hom}_K^{2 \rightarrow 2}$ gives a linear bijection

$$(2.3) \quad \tilde{\Gamma}_0^V : \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{F}(G/K)) \xrightarrow{\sim} \text{Hom}_W(V_{\text{single}}^M, \mathcal{F}(A)).$$

The following lemma will be used repeatedly to deduce generalized radial part formulas:

Lemma 2.7. Suppose $V \in \widehat{K}_M$ and let p^V be as in Theorem 2.3 (ii). Suppose $\alpha \in \Sigma$ and $X_\alpha \in \mathfrak{g}_\alpha$.

(i) $(X_\alpha + \theta X_\alpha)^2 v = |\alpha|^2 B(X_\alpha, \theta X_\alpha)(1 - s_\alpha)v$ for $v \in V_{\text{single}}^M$, where $|\alpha| = \sqrt{B(H_\alpha, H_\alpha)}$ and $s_\alpha \in W$ is the reflection corresponding to α .

(ii) $p^V((X_\alpha + \theta X_\alpha)^2 v) \in V_{\text{double}}^M$ for $v \in V_{\text{double}}^M$.

Proof. (i) is only a restatement of [O, (3.10)]. To show (ii) let (\cdot, \cdot) be a K -invariant inner product of V . Then the proof of [O, Lemma 3.3] shows $V_{\text{single}}^M \perp V_{\text{double}}^M$ with respect to (\cdot, \cdot) . If $v_1 \in V_{\text{single}}^M$ and $v_2 \in V_{\text{double}}^M$ then

$$\begin{aligned}(v_1, p^V((X_\alpha + \theta X_\alpha)^2 v_2)) &= (v_1, (X_\alpha + \theta X_\alpha)^2 v_2) \\ &= ((X_\alpha + \theta X_\alpha)^2 v_1, v_2) \\ &= |\alpha|^2 B(X_\alpha, \theta X_\alpha)((1 - s_\alpha)v_1, v_2) \\ &= 0.\end{aligned}$$

Hence $p^V((X_\alpha + \theta X_\alpha)^2 v_2) \in V^M \cap (V_{\text{single}}^M)^\perp = V_{\text{double}}^M$. \square

3. DUNKL OPERATORS

Put $R := 2\Sigma$, $R^+ := 2\Sigma^+$, $R_1 := R \setminus 2R$, and $R_1^+ := R_1 \cap R^+$. Then R and R_1 are root systems sharing the same Weyl group W with Σ . In [Dun] Dunkl introduces the following operators:

Definition 3.1 (Dunkl operators). Suppose $\mathbf{k} : W \setminus R_1 \rightarrow \mathbb{C}$ is a *multiplicity function*, namely, a \mathbb{C} -valued function on R_1 which is constant on each W -orbit. For $\xi \in \mathfrak{a}$ define $\mathcal{D}_{\mathbf{k}}(\xi) \in \text{End}_{\mathbb{C}} C^\infty(\mathfrak{a})$ by

$$\mathcal{D}_{\mathbf{k}}(\xi) = \partial(\xi) + \sum_{\alpha \in R_1^+} \mathbf{k}(\alpha) \frac{\alpha(\xi)}{\alpha} (1 - s_\alpha)$$

where $\partial(\xi)$ is the ξ -directional derivative operator. When \mathbf{k} is a special multiplicity function $\mathbf{m}_0 : W \setminus R_1 \rightarrow \mathbb{C}$ specified by

$$\mathbf{m}_0(\alpha) = \begin{cases} \frac{1}{2} \dim \mathfrak{g}_{\alpha/2} & \text{if } 2\alpha \notin R, \\ \frac{1}{2} (\dim \mathfrak{g}_{\alpha/2} + \dim \mathfrak{g}_\alpha) & \text{if } 2\alpha \in R, \end{cases}$$

we use the brief symbol \mathcal{D} for $\mathcal{D}_{\mathbf{m}_0}$.

Remark 3.2. There is no significant meaning in using R or R_1 instead of Σ . We do so only for the compatibility with the case of Cherednik operators (Definition 4.2).

The following are well-known properties of Dunkl operators:

Proposition 3.3 ([Dun]). (i) For $\xi, \eta \in \mathfrak{a}$, $\mathcal{D}_{\mathbf{k}}(\xi)\mathcal{D}_{\mathbf{k}}(\eta) = \mathcal{D}_{\mathbf{k}}(\eta)\mathcal{D}_{\mathbf{k}}(\xi)$.
(ii) For $\xi \in \mathfrak{a}$ and $w \in W$, $w\mathcal{D}_{\mathbf{k}}(\xi)w^{-1} = \mathcal{D}_{\mathbf{k}}(w\xi)$.
(iii) Let $\{\xi_1, \dots, \xi_\ell\}$ be an orthonormal basis of the Euclidean space $(\mathfrak{a}, B(\cdot, \cdot))$ and put $L_{\mathfrak{a}} = \sum_{i=1}^{\ell} \xi_i^2 \in S(\mathfrak{a}_{\mathbb{C}})$. Then

$$\mathcal{D}_{\mathbf{k}}(L_{\mathfrak{a}}) = \partial(L_{\mathfrak{a}}) + \sum_{\alpha \in R_1^+} \mathbf{k}(\alpha) \left(\frac{2}{\alpha} \partial(H_\alpha) - \frac{|\alpha|^2}{\alpha^2} (1 - s_\alpha) \right).$$

Now let us state a generalization of (1.16) for $f \notin C^\infty(\mathfrak{s})^K$.

Theorem 3.4 (the radial part formula). Suppose $V \in \widehat{K}_M$, $\Delta \in S(\mathfrak{s}_{\mathbb{C}})^K$ and $\Phi \in \text{Hom}_K(V, C^\infty(\mathfrak{s}))$.

(i) We have

$$\begin{aligned} \partial(\Delta) \circ \Phi &\in \text{Hom}_K(V, C^\infty(\mathfrak{s})), \\ \mathcal{D}(\gamma_0(\Delta)) \circ \Gamma_0^V(\Phi) &\in \text{Hom}_W(V^M, C^\infty(\mathfrak{a})), \\ \mathcal{D}(\gamma_0(\Delta)) \circ \tilde{\Gamma}_0^V(\Phi) &\in \text{Hom}_W(V_{\text{single}}^M, C^\infty(\mathfrak{a})), \end{aligned}$$

and it holds that

$$(3.1) \quad \tilde{\Gamma}_0^V(\partial(\Delta) \circ \Phi) = \mathcal{D}(\gamma_0(\Delta)) \circ \tilde{\Gamma}_0^V(\Phi).$$

(ii) If $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$ then $\partial(\Delta) \circ \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$. Hence for such Φ

$$(3.2) \quad \Gamma_0^V(\partial(\Delta) \circ \Phi) = \mathcal{D}(\gamma_0(\Delta)) \circ \Gamma_0^V(\Phi).$$

(iii) If $V \in \widehat{K}_{\text{sp}}$ then (3.2) always holds.

On the other hand, as a generalization of (1.16) for $\Delta \notin S(\mathfrak{s}_{\mathbb{C}})^K$ we have

Theorem 3.5 (the radial part formula). *Suppose $V \in \widehat{K}_M$, $\Phi \in \text{Hom}_K(V, S(\mathfrak{s}_{\mathbb{C}}))$ and $f \in C^\infty(\mathfrak{s})^K$. Then $\Gamma_0^V(\Phi) \in \text{Hom}_W(V^M, S(\mathfrak{a}_{\mathbb{C}}))$ is naturally defined by the identifications $S(\mathfrak{s}_{\mathbb{C}}) \simeq \mathcal{P}(\mathfrak{s})$ and $S(\mathfrak{a}_{\mathbb{C}}) \simeq \mathcal{P}(\mathfrak{a})$. Let $\partial(\Phi)f$ denote the map*

$$V \ni v \longmapsto \partial(\Phi[v])f \in C^\infty(\mathfrak{s}).$$

Let $\mathcal{D}(\Gamma_0^V(\Phi))\gamma_0(f)$ denote the map

$$V^M \ni v \longmapsto \mathcal{D}(\Gamma_0^V(\Phi)[v])\gamma_0(f) \in C^\infty(\mathfrak{a}).$$

(i) We have

$$\begin{aligned} \partial(\Phi)f &\in \text{Hom}_K(V, C^\infty(\mathfrak{s})), \\ \mathcal{D}(\Gamma_0^V(\Phi))\gamma_0(f) &\in \text{Hom}_W(V^M, C^\infty(\mathfrak{a})), \end{aligned}$$

and it holds that

$$(3.3) \quad \Gamma_0^V(\partial(\Phi)f)[v] = \mathcal{D}(\Gamma_0^V(\Phi))\gamma_0(f)[v]. \quad \text{for } v \in V_{\text{single}}^M.$$

(ii) If $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, S(\mathfrak{s}_{\mathbb{C}}))$ ($:= \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{P}(\mathfrak{s}))$), then $\partial(\Phi)f \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$. Hence for such Φ

$$(3.4) \quad \Gamma_0^V(\partial(\Phi)f) = \mathcal{D}(\Gamma_0^V(\Phi))\gamma_0(f).$$

(iii) If $V \in \widehat{K}_{\text{sp}}$ then (3.4) always holds.

From now on we shall prove these theorems by following the method of de Jeu [Je], which uses only three simple lemmas.

Lemma 3.6. *Let $\{X_1, \dots, X_{\dim \mathfrak{s}}\}$ be an orthonormal basis of the Euclidean space $(\mathfrak{s}, B(\cdot, \cdot))$ and put $L_{\mathfrak{s}} = \sum_{i=1}^{\dim \mathfrak{s}} X_i^2 \in S(\mathfrak{s}_{\mathbb{C}})$. Then Theorem 3.4 for $\Delta = L_{\mathfrak{s}}$ is true.*

Proof. Suppose $V \in \widehat{K}_M$ and $\Phi \in \text{Hom}_K(V, C^\infty(\mathfrak{s}))$. The first assertion of Theorem 3.4 (i) is trivial. Although (3.1) for $\Delta = L_{\mathfrak{s}}$ is equivalent to [O, Lemma 3.10], we recall the outline of its proof. For each $\alpha \in \Sigma^+$ choose an orthonormal basis $\{X_\alpha^{(1)}, \dots, X_\alpha^{(\dim \mathfrak{g}_\alpha)}\}$ of the Euclidean space $(\mathfrak{g}_\alpha, -B(\cdot, \theta \cdot))$. Let $v \in V^M$ and $H \in \mathfrak{a}$. Then a direct calculation (cf. the proof of [O, Lemma 3.10]) leads to

$$(3.5) \quad \begin{aligned} \partial(L_{\mathfrak{s}})\Phi[v](H) &= \partial(L_{\mathfrak{a}})\Phi[v](H) + \sum_{\alpha \in R_1^+} \frac{2\mathbf{m}_0(\alpha)}{\alpha(H)} \partial(H_\alpha)\Phi[v](H) \\ &\quad + \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{\dim \mathfrak{g}_\alpha} \frac{\Phi \left[p^V \left((X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 v \right) \right](H)}{2\alpha(H)^2}. \end{aligned}$$

If $v \in V_{\text{single}}^M$, then Lemma 2.7 (i) reduces the right-hand side of (3.5) to

$$\partial(L_{\mathfrak{a}})\Phi[v](H) + \sum_{\alpha \in R_1^+} \mathbf{m}_0(\alpha) \left(\frac{2}{\alpha(H)} \partial(H_\alpha)\Phi[v](H) - |\alpha|^2 \frac{\Phi[v](H) - \Phi[v](s_\alpha H)}{\alpha(H)^2} \right),$$

which equals $\mathcal{D}(L_{\mathfrak{a}})\Phi[v](H)$ by Proposition 3.3 (iii). Since $\gamma_0(L_{\mathfrak{s}}) = L_{\mathfrak{a}}$, we get (3.1) for $\Delta = L_{\mathfrak{s}}$.

In order to show (ii) for $\Delta = L_{\mathfrak{s}}$, suppose $\Gamma_0^V(\Phi)[V_{\text{double}}^M] = \{0\}$. Then by virtue of Lemma 2.7 (ii) the right-hand side of (3.5) is 0 for $v \in V_{\text{double}}^M$. This means $\Gamma_0^V(\partial(L_{\mathfrak{s}}) \circ \Phi)[V_{\text{double}}^M] = \{0\}$, proving (ii) for $\Delta = L_{\mathfrak{s}}$.

Finally (iii) is immediate from (i) or (ii) (cf. Remark 2.6 (i)). \square

Lemma 3.7 ([Hec1, Je]). *Suppose p is a homogeneous element of $S(\mathfrak{a}_{\mathbb{C}})$ with degree d . Identifying p with a polynomial function on \mathfrak{a} , let $\underline{p} \in \text{End}_{\mathbb{C}} C^\infty(\mathfrak{a})$ denote the multiplication operator by p . Then in the algebra $\text{End}_{\mathbb{C}} C^\infty(\mathfrak{a})$ the following identity holds:*

$$\frac{1}{d!} \left(\text{ad} \frac{\mathcal{D}_{\mathbf{k}}(L_{\mathfrak{a}})}{2} \right)^d \underline{p} = \mathcal{D}_{\mathbf{k}}(p).$$

Lemma 3.8. *Suppose P is a homogeneous element of $S(\mathfrak{s}_{\mathbb{C}})$ with degree d . Identifying P with a polynomial function on \mathfrak{s} , let $\underline{P} \in \text{End}_{\mathbb{C}} C^\infty(\mathfrak{s})$ denote the multiplication operator by P . Then in the algebra $\text{End}_{\mathbb{C}} C^\infty(\mathfrak{s})$ the following identity holds:*

$$\frac{1}{d!} \left(\text{ad} \frac{\partial(L_{\mathfrak{s}})}{2} \right)^d \underline{P} = \partial(P).$$

Proof. By an elementary calculation. \square

Proof of Theorem 3.4. Suppose $V \in \widehat{K}_M$, $\Delta \in S(\mathfrak{s}_{\mathbb{C}})^K$, and $\Phi \in \text{Hom}_K(V, C^\infty(\mathfrak{s}))$. We may assume Δ is homogeneous with degree d . By Lemma 3.8 we have for $v \in V$

$$(3.6) \quad \partial(\Delta)\Phi[v] = \frac{1}{d!} \left(\left(\text{ad} \frac{\partial(L_{\mathfrak{s}})}{2} \right)^d \underline{\Delta} \right) \Phi[v].$$

Now let $v \in V_{\text{single}}^M$. Then restricting both sides of (3.6) to \mathfrak{a} we have

$$\begin{aligned} \gamma_0(\partial(\Delta)\Phi[v]) &= \frac{1}{d!} \left(\left(\text{ad} \frac{\mathcal{D}(L_{\mathfrak{a}})}{2} \right)^d \underline{\gamma_0(\Delta)} \right) \gamma_0(\Phi[v]) & (\because \text{Lemma 3.6}) \\ &= \mathcal{D}(\gamma_0(\Delta))\gamma_0(\Phi[v]), & (\because \text{Lemma 3.7}) \end{aligned}$$

which proves Theorem 3.4 (i) and hence (iii).

In general, if $\Psi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$, then $\partial(L_{\mathfrak{s}}) \circ \Psi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$ by Lemma 3.6 and clearly $\underline{\Delta} \circ \Psi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$. Hence it follows from (3.6) $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$ implies $\partial(\Delta) \circ \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$. We thus get Theorem 3.4 (ii). \square

Proof of Theorem 3.5. Suppose $V \in \widehat{K}_M$, $\Phi \in \text{Hom}_K(V, S(\mathfrak{s}_{\mathbb{C}}))$, and $f \in C^\infty(\mathfrak{s})^K$. We may assume $\Phi[v]$ for any $v \in V$ is a homogeneous element with a constant degree d . The first assertion of Theorem 3.5 (i) is clear. By Lemma 3.8 we have for $v \in V$

$$(3.7) \quad \begin{aligned} \partial(\Phi[v])f &= \frac{1}{d!} \left(\left(\text{ad} \frac{\partial(L_{\mathfrak{s}})}{2} \right)^d \Phi[v] \right) f \\ &= \sum_{d_1=0}^d \frac{(-1)^{d_1}}{(d-d_1)!d_1!} \left(\frac{\partial(L_{\mathfrak{s}})}{2} \right)^{d-d_1} \underline{\Phi[v]} \left(\frac{\partial(L_{\mathfrak{s}})}{2} \right)^{d_1} f. \end{aligned}$$

For $d_1 = 0, \dots, d$ define $\Psi_{d_1} \in \text{Hom}_K(V, C^\infty(\mathfrak{s}))$ by

$$V \ni v \mapsto \underline{\Phi[v]} \left(\frac{\partial(L_{\mathfrak{s}})}{2} \right)^{d_1} f.$$

Then by Lemma 3.6 with $V = \mathbb{C}_{\text{triv}}$ (the trivial K -type) we have

$$(3.8) \quad \Gamma_0^V(\Psi_{d_1})[v] = \underline{\Gamma_0^V(\Phi)[v]} \left(\frac{\mathcal{D}(L_{\mathfrak{a}})}{2} \right)^{d_1} \gamma_0(f) \quad \text{for } v \in V^M.$$

Now if $v \in V_{\text{single}}^M$ then the restriction of (3.7) is calculated to be

$$\begin{aligned} & \gamma_0(\partial(\Phi[v])f) \\ &= \sum_{d_1=0}^d \frac{(-1)^{d_1}}{(d-d_1)!d_1!} \left(\frac{\mathcal{D}(L_{\mathfrak{a}})}{2} \right)^{d-d_1} \Gamma_0^V(\Psi_{d_1})[v] \quad (\because \text{Lemma 3.6}) \\ &= \sum_{d_1=0}^d \frac{(-1)^{d_1}}{(d-d_1)!d_1!} \left(\frac{\mathcal{D}(L_{\mathfrak{a}})}{2} \right)^{d-d_1} \underline{\Gamma_0^V(\Phi)[v]} \left(\frac{\mathcal{D}(L_{\mathfrak{a}})}{2} \right)^{d_1} \gamma_0(f) \quad (\because (3.8)) \\ &= \frac{1}{d!} \left(\left(\text{ad } \frac{\mathcal{D}(L_{\mathfrak{a}})}{2} \right)^d \underline{\Gamma_0^V(\Phi)[v]} \right) \gamma_0(f) \\ &= \mathcal{D}(\Gamma_0^V(\Phi)[v]) \gamma_0(f). \quad (\because \text{Lemma 3.7}) \end{aligned}$$

It proves (3.3) and therefore (iii).

In order to show (ii) suppose $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, S(\mathfrak{s}_{\mathbb{C}}))$. Then by (3.8) we have $\Psi_{d_1} \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$ for $d_1 = 0, \dots, d$. Since

$$\partial(\Phi)f = \sum_{d_1=0}^d \frac{(-1)^{d_1}}{(d-d_1)!d_1!} \left(\frac{\partial(L_{\mathfrak{s}})}{2} \right)^{d-d_1} \circ \Psi_{d_1},$$

it follows from Lemma 3.6 that $\partial(\Phi)f \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{s}))$. We thus get (ii). \square

4. GRADED HECKE ALGEBRAS AND CHEREDNIK OPERATORS

Let Π be the system of simple roots in $R_1 = 2\Sigma \setminus 4\Sigma$ corresponding to the positive system $R_1^+ = 2\Sigma^+ \setminus 4\Sigma^+$.

Definition 4.1 (graded Hecke algebras [Lu]). Let $\mathbf{k} : W \setminus R_1 \rightarrow \mathbb{C}$ be a multiplicity function. Then there exists uniquely (up to equivalence) an algebra $\mathbf{H}_{\mathbf{k}}$ over \mathbb{C} with the following properties:

- (i) $\mathbf{H}_{\mathbf{k}} \simeq S(\mathfrak{a}_{\mathbb{C}}) \otimes \mathbb{C}W$ as a \mathbb{C} -linear space;
- (ii) The maps $S(\mathfrak{a}_{\mathbb{C}}) \rightarrow \mathbf{H}_{\mathbf{k}}, \varphi \mapsto \varphi \otimes 1$ and $\mathbb{C}W \rightarrow \mathbf{H}_{\mathbf{k}}, w \mapsto 1 \otimes w$ are algebra homomorphisms;
- (iii) $(\varphi \otimes 1) \cdot (1 \otimes w) = \varphi \otimes w$ for any $\varphi \in S(\mathfrak{a}_{\mathbb{C}})$ and $w \in W$;
- (iv) $(1 \otimes s_{\alpha}) \cdot (\xi \otimes 1) = s_{\alpha}(\xi) \otimes s_{\alpha} - \mathbf{k}(\alpha) \alpha(\xi)$ for any $\alpha \in \Pi$ and $\xi \in \mathfrak{a}_{\mathbb{C}}$.

We call $\mathbf{H}_{\mathbf{k}}$ the graded Hecke algebra associated to the data $(\mathfrak{a}, \Pi, \mathbf{k})$.

By (ii) we identify $S(\mathfrak{a}_{\mathbb{C}})$ and $\mathbb{C}W$ with subalgebras of $\mathbf{H}_{\mathbf{k}}$. Then (iv) is simply written as

$$(4.1) \quad s_{\alpha} \cdot \xi = s_{\alpha}(\xi) \cdot s_{\alpha} - \mathbf{k}(\alpha) \alpha(\xi) \quad \forall \alpha \in \Pi \quad \forall \xi \in \mathfrak{a}_{\mathbb{C}}.$$

It is well known that the center of $\mathbf{H}_{\mathbf{k}}$ equals $S(\mathfrak{a}_{\mathbb{C}})^W$ (cf. [Lu, Theorem 6.5]). Define the multiplicity function $\mathbf{m}_1 : W \backslash R_1 \rightarrow \mathbb{C}$ by

$$\mathbf{m}_1(\alpha) = \begin{cases} \frac{1}{2} \dim \mathfrak{g}_{\alpha/2} & \text{if } 2\alpha \notin R, \\ \frac{1}{2} \dim \mathfrak{g}_{\alpha/2} + \dim \mathfrak{g}_{\alpha} & \text{if } 2\alpha \in R, \end{cases}$$

and put $\mathbf{H} = \mathbf{H}_{\mathbf{m}_1}$. We consider \mathbf{H} is a special graded Hecke algebra attached to the Iwasawa decomposition $G = NAK$.

Now recall $R = 2\Sigma$ and $R^+ = 2\Sigma^+$. By (1.15) we identify $C^{\infty}(\mathfrak{a})$ with $C^{\infty}(A)$, so that $\text{End}_{\mathbb{C}} C^{\infty}(A) \ni \partial(\xi)$ ($\xi \in \mathfrak{a}_{\mathbb{C}}$). For $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ let $e^{\mu} \in C^{\infty}(A)$ denote the function $a \mapsto \exp \mu(\log a)$.

Definition 4.2 (Cherednik operators [Ch1]). Suppose $\mathbf{k} : W \backslash R \rightarrow \mathbb{C}$ is a multiplicity function and put $\rho_{\mathbf{k}} = \frac{1}{2} \sum_{\alpha \in R^+} \mathbf{k}(\alpha) \alpha$. For $\xi \in \mathfrak{a}$ define $\mathcal{T}_{\mathbf{k}}(\xi) \in \text{End}_{\mathbb{C}} C^{\infty}(A)$ by

$$\mathcal{T}_{\mathbf{k}}(\xi) = \partial(\xi) + \sum_{\alpha \in R^+} \mathbf{k}(\alpha) \frac{\alpha(\xi)}{1 - e^{-\alpha}} (1 - s_{\alpha}) - \rho_{\mathbf{k}}(\xi).$$

If \mathbf{k} equals the multiplicity function $\mathbf{m} : W \backslash R \rightarrow \mathbb{C}$ defined by

$$\mathbf{m}(\alpha) = \frac{1}{2} \dim \mathfrak{g}_{\alpha/2},$$

then we use the brief symbol \mathcal{T} for $\mathcal{T}_{\mathbf{m}}$.

The following are fundamental properties of Cherednik operators:

Proposition 4.3. (i) For $\xi, \eta \in \mathfrak{a}$, $\mathcal{T}_{\mathbf{k}}(\xi) \mathcal{T}_{\mathbf{k}}(\eta) = \mathcal{T}_{\mathbf{k}}(\eta) \mathcal{T}_{\mathbf{k}}(\xi)$.
(ii) Let $\mathbf{k}_1 : W \backslash R_1 \rightarrow \mathbb{C}$ be the multiplicity function defined by

$$\mathbf{k}_1(\alpha) = \begin{cases} \mathbf{k}(\alpha) & \text{if } 2\alpha \notin R, \\ \mathbf{k}(\alpha) + 2\mathbf{k}(2\alpha) & \text{if } 2\alpha \in R. \end{cases}$$

Then $s_{\alpha} \mathcal{T}_{\mathbf{k}}(\xi) = \mathcal{T}_{\mathbf{k}}(s_{\alpha}(\xi)) s_{\alpha} - \mathbf{k}_1(\alpha) \alpha(\xi)$ for $\xi \in \mathfrak{a}$ and $\alpha \in \Pi$. Hence $\mathcal{T}_{\mathbf{k}} : \mathfrak{a} \rightarrow \text{End}_{\mathbb{C}} C^{\infty}(A)$ uniquely extends to an algebra homomorphism of $\mathbf{H}_{\mathbf{k}_1}$ into $\text{End}_{\mathbb{C}} C^{\infty}(A)$.

(iii) Let $L_{\mathfrak{a}} \in S(\mathfrak{a}_{\mathbb{C}})$ be as in Proposition 3.3 (iii). Then

$$(4.2) \quad \mathcal{T}_{\mathbf{k}}(L_{\mathfrak{a}}) = \partial(L_{\mathfrak{a}}) + \sum_{\alpha \in R^+} \mathbf{k}(\alpha) \left(\coth \frac{\alpha}{2} \partial(H_{\alpha}) - \frac{|\alpha|^2}{4 \sinh^2 \frac{\alpha}{2}} (1 - s_{\alpha}) \right) + B(H_{\rho_{\mathbf{k}}}, H_{\rho_{\mathbf{k}}}).$$

Proof. Proposition 4.3 (i), (ii) are given in [Ch1] (see also [Op1]). (iii) is calculated in [Sha]. \square

Remark 4.4. If $\mathbf{k} = \mathbf{m}$ then $\rho_{\mathbf{k}} = \rho$, $\mathbf{k}_1 = \mathbf{m}_1$ and $\mathbf{H}_{\mathbf{k}_1} = \mathbf{H}$.

5. RADIAL PART FORMULA, I

The next theorem is a generalization of (1.3) to some cases where f is no longer K -invariant. The tangential counterpart of this theorem is Theorem 3.4.

Theorem 5.1 (the radial part formula). *Suppose $V \in \widehat{K}_M$, $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K$, and $\Phi \in \text{Hom}_K(V, C^\infty(G/K))$.*

(i) *We have*

$$\begin{aligned} r(\Delta) \circ \Phi &\in \text{Hom}_K(V, C^\infty(G/K)), \\ \mathcal{T}(\gamma(\Delta)) \circ \Gamma_0^V(\Phi) &\in \text{Hom}_W(V^M, C^\infty(A)), \\ \mathcal{T}(\gamma(\Delta)) \circ \tilde{\Gamma}_0^V(\Phi) &\in \text{Hom}_W(V_{\text{single}}^M, C^\infty(A)), \end{aligned}$$

and it holds that

$$(5.1) \quad \tilde{\Gamma}_0^V(r(\Delta) \circ \Phi) = \mathcal{T}(\gamma(\Delta)) \circ \tilde{\Gamma}_0^V(\Phi).$$

(ii) *If $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(G/K))$ then $r(\Delta) \circ \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(G/K))$. Hence for such Φ*

$$(5.2) \quad \Gamma_0^V(r(\Delta) \circ \Phi) = \mathcal{T}(\gamma(\Delta)) \circ \Gamma_0^V(\Phi).$$

(iii) *If $V \in \widehat{K}_{\text{sp}}$ then (5.2) always holds.*

Corollary 5.2. *Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. For each $V \in \widehat{K}_M$ put*

$$\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{A}(G/K, \lambda)) = \text{Hom}_K(V, \mathcal{A}(G/K, \lambda)) \cap \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{C}^\infty(G/K)).$$

Then $\tilde{\Gamma}_0^V$ induces a linear bijection

$$(5.3) \quad \tilde{\Gamma}_0^V : \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{A}(G/K, \lambda)) \xrightarrow{\sim} \text{Hom}_W(V_{\text{single}}^M, \mathcal{A}(A, \lambda)).$$

In particular, if $V \in \widehat{K}_{\text{sp}}$ then (1.9) holds.

Proof. By Remark 2.6 (ii), (1.7), and the above theorem. \square

The rest of this section is devoted to the proof of Theorem 5.1. Let \mathcal{R} be the algebra of those power series of $\{e^\alpha; \alpha \in \Sigma^+\}$ which absolutely converge on

$$A_- := \{e^H; H \in \mathfrak{a} \text{ with } \alpha(H) < 0 \text{ for any } \alpha \in \Sigma^+\}.$$

Each element $c \in \mathcal{R}$ is uniquely expanded as

$$c = \sum_{\lambda \in \mathbb{Z}_{\geq 0} \Sigma^+} c_\lambda e^\lambda \quad \text{with } c_\lambda \in \mathbb{C}.$$

Using this expansion we put

$$\text{spec } c = \{\lambda; c_\lambda \neq 0\}.$$

This is a subset of $\mathbb{Z}_{\geq 0} \Sigma^+$. In the argument below, the maximal ideal

$$\mathcal{M} := \{c \in \mathcal{R}; \text{spec } c \not\ni 0\}$$

has important roles.

Suppose $x \in A_-$ and put $\bar{x} = xK \in G/K$. Let C_{wx}^∞ ($w \in W$) be the space of germs of C^∞ functions at wx , that is

$$C_{wx}^\infty = \varinjlim_{\mathcal{U}; wx \in \mathcal{U} \subset A, \text{ open}} C^\infty(\mathcal{U}).$$

If we put $C^\infty(Wx) = \bigoplus_{w \in W} C_{wx}^\infty$, then W naturally acts on $C^\infty(Wx)$. Since x is a regular point of A , for any W -module U

$$(5.4) \quad \text{Hom}_W(U, C^\infty(Wx)) \xrightarrow{\sim} \text{Hom}_\mathbb{C}(U, C_x^\infty)$$

by restriction. Let us think of $\mathcal{R} \otimes \partial(S(\mathfrak{a}_\mathbb{C})) \otimes \text{End}_\mathbb{C} U^*$ as the tensor product of the ring $\mathcal{R} \otimes \partial(S(\mathfrak{a}_\mathbb{C}))$ of differential operators on A_- with coefficients in \mathcal{R} and the endomorphism ring $\text{End}_\mathbb{C} U^*$ for the dual space U^* of U . (In this paper \otimes is taken over \mathbb{C} unless otherwise specified.) Then $\mathcal{R} \otimes \partial(S(\mathfrak{a}_\mathbb{C})) \otimes \text{End}_\mathbb{C} U^*$ acts on $\varphi \in \text{Hom}_\mathbb{C}(U, C_x^\infty)$ by

$$((D \otimes \tau)\varphi)[u] = D(\varphi[{}^t\tau(u)]) \quad \text{with } D \in \mathcal{R} \otimes \partial(S(\mathfrak{a}_\mathbb{C})), \tau \in \text{End}_\mathbb{C} U^*, u \in U,$$

where ${}^t\tau \in \text{End}_\mathbb{C} U$ is the transpose of τ .

Now, if we put

$$C^\infty(K\bar{x}) := \varinjlim_{\mathcal{V} \supset K\bar{x}; \text{ open}} C^\infty(\mathcal{V}),$$

then $\ell(U(\mathfrak{g}_\mathbb{C}))$, $\ell(K)$ and $r(U(\mathfrak{g}_\mathbb{C})^K)$ naturally act on it. For any $V \in \widehat{K}_M$ define a localized restriction map

$$\begin{aligned} \Gamma_{0,Wx}^V : \text{Hom}_K(V, C^\infty(K\bar{x})) &\longrightarrow \text{Hom}_W(V^M, C^\infty(Wx)) \left(\simeq \text{Hom}_\mathbb{C}(V^M, C_x^\infty) \right); \\ \Phi &\longmapsto \left(\varphi : V^M \hookrightarrow V \xrightarrow{\Phi} C^\infty(K\bar{x}) \xrightarrow{\gamma_0} C^\infty(Wx) \text{ (or } C_x^\infty \text{)} \right). \end{aligned}$$

When we consider the target space is $\text{Hom}_\mathbb{C}(V^M, C_x^\infty)$, we use the symbol $\Gamma_{0,x}^V$ instead of $\Gamma_{0,Wx}^V$.

Lemma 5.3. $\Gamma_{0,x}^V$ (or equivalently $\Gamma_{0,Wx}^V$) is a bijection.

Proof. Let $p^V : V \rightarrow V^M$ be as in Theorem 2.3 (ii). For any open neighbourhood $\mathcal{V} \subset G/K$ of $K\bar{x}$, we can take a sufficiently small open neighbourhood $\mathcal{U} \subset A_-$ of x so that $K\mathcal{U} \simeq K/M \times \mathcal{U} \subset \mathcal{V}$. If $\Phi \in \text{Hom}_K(V, C^\infty(K\mathcal{U}))$ then for $v \in V$ and $(kM, y) \in K/M \times \mathcal{U}$ we have

$$\Phi[v](ky) = \Phi[k^{-1}v](y) = \Phi[p^V(k^{-1}v)](y) = \Gamma_{0,\mathcal{U}}^V(\Phi)[p^V(k^{-1}v)](y).$$

This shows the injectivity of $\Gamma_{0,x}^V$. Conversely, for any $\varphi \in \text{Hom}_\mathbb{C}(V^M, C^\infty(\mathcal{U}))$,

$$\Phi[v](ky) := \varphi[p^V(k^{-1}v)](y) \quad (v \in V, (kM, y) \in K/M \times \mathcal{U})$$

defines its lift. This shows the surjectivity. \square

Lemma 5.4. Suppose $V \in \widehat{K}_M$. For any $\Delta \in U(\mathfrak{g}_\mathbb{C})^K$ there exists a unique $E \in \mathcal{M} \otimes \partial(S(\mathfrak{a}_\mathbb{C})) \otimes \text{End}_\mathbb{C}(V^M)^*$ such that

$$\Gamma_{0,x}^V(r(\Delta) \circ \Phi) = (\partial(\gamma(\Delta))(\cdot - \rho) + E) \Gamma_{0,x}^V(\Phi) \quad \text{for any } \Phi \in \text{Hom}_K(V, C^\infty(K\bar{x})).$$

Remark 5.5. (i) $\mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}}))$ is an ideal of $\mathcal{R} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}}))$.

(ii) $\gamma(\Delta)(\cdot - \rho)$ is nothing but the second summand of Δ in the direct sum decomposition $U(\mathfrak{g}_{\mathbb{C}}) = (\mathfrak{n}_{\mathbb{C}}U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}) \oplus U(\mathfrak{a}_{\mathbb{C}})$.

(iii) Let $\{v_1, \dots, v_{m'}\}$ and $\{v_{m'+1}, \dots, v_m\}$ be bases of V_{single}^M and V_{double}^M respectively. Then $\{v_1, \dots, v_m\}$ is a basis of V^M . Let $\{v_1^*, \dots, v_m^*\} \subset (V^M)^*$ be the dual basis of $\{v_1, \dots, v_m\}$. With respect to these bases, we can express any element of $\mathcal{R} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})) \otimes \text{End}_{\mathbb{C}}(V^M)^*$ in a matrix form. More precisely, the correspondence

$$\sum_{i,j} D_{ij} \otimes (v_i^* \otimes v_j) \mapsto (D_{ij})$$

gives an algebra isomorphism $\mathcal{R} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})) \otimes \text{End}_{\mathbb{C}}(V^M)^* \xrightarrow{\sim} \text{Mat}(m, m; \mathcal{R} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$. Moreover, if we identify $\varphi \in \text{Hom}_{\mathbb{C}}(V^M, C_x^\infty)$ with a column vector ${}^t(\varphi[v_1], \dots, \varphi[v_m]) \in (C_x^\infty)^m$, then the action of $\mathcal{R} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})) \otimes \text{End}_{\mathbb{C}}(V^M)^*$ reduces to the left multiplication. In the proof of Theorem 5.1 we use the following matrix expression of E :

$$E = \begin{pmatrix} E_{\text{single}} & P \\ Q & E_{\text{double}} \end{pmatrix}.$$

Here the matrix is divided into four blocks according to the division of the basis

$$\{v_1, \dots, v_m\} = \{v_1, \dots, v_{m'}\} \sqcup \{v_{m'+1}, \dots, v_m\}.$$

Proof of Lemma 5.4. Suppose $X_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Sigma^+$). For any $y = e^H \in A_-$ we have

$$\begin{aligned} \text{Ad}(y^{-1})(X_\alpha + \theta X_\alpha) &= e^{-\alpha(H)} X_\alpha + e^{\alpha(H)} \theta X_\alpha \\ &= (e^{-\alpha(H)} - e^{\alpha(H)}) X_\alpha + e^{\alpha(H)} (X_\alpha + \theta X_\alpha) \end{aligned}$$

and therefore

$$X_\alpha = \frac{e^{\alpha(H)}}{1 - e^{2\alpha(H)}} \text{Ad}(y^{-1})(X_\alpha + \theta X_\alpha) - \frac{e^{2\alpha(H)}}{1 - e^{2\alpha(H)}} (X_\alpha + \theta X_\alpha).$$

Hence if $D \in U(\mathfrak{g}_{\mathbb{C}})$ is given, we can take $c_i \in \mathcal{M}$, $D'_i \in U(\mathfrak{k}_{\mathbb{C}})$, and $D''_i \in S(\mathfrak{a}_{\mathbb{C}})$ ($i = 1, \dots, q$) so that it holds that

$$(5.5) \quad D \equiv \gamma(D)(\cdot - \rho) + \sum_{i=1}^q c_i(y) \text{Ad}(y^{-1})(D'_i) D''_i \pmod{U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}}$$

for any $y \in A_-$. (This is shown by induction on the order of D . A more detailed argument can be found in the proof of [Hel4, Ch. II, Proposition 5.23].) Applying (5.5) to $D = \Delta$, we have for any $\Phi \in \text{Hom}_K(V, C^\infty(K\bar{x}))$ and $v \in V^M$

$$\gamma_{0,x}(r(\Delta)\Phi[v]) = \partial(\gamma(\Delta)(\cdot - \rho))\gamma_{0,x}(\Phi[v]) + \sum_{i=1}^q c_i \partial(D''_i) \gamma_{0,x}(\ell({}^t D'_i)\Phi[v]),$$

where $\gamma_{0,x}$ is the restriction map $C^\infty(K\bar{x}) \rightarrow C_x^\infty$ and ${}^t \cdot$ is the anti-automorphism of $U(\mathfrak{g}_{\mathbb{C}})$ defined by ${}^t X = -X$ for $X \in \mathfrak{g}_{\mathbb{C}}$. Let π_V be the $U(\mathfrak{k}_{\mathbb{C}})$ -action on V . Since $\gamma_{0,x} \circ \Phi = \gamma_{0,x} \circ \Phi \circ p^V$, we have for $i = 1, \dots, q$

$$\gamma_{0,x}(\ell({}^t D'_i)\Phi[v]) = \gamma_{0,x}(\Phi[\pi_V({}^t D'_i)v]) = \gamma_{0,x}(\Phi[p^V \circ \pi_V({}^t D'_i)v]).$$

Hence we can take

$$E = \sum_{i=1}^q c_i \otimes \partial(D_i'') \otimes {}^t(p^V \circ \pi_V({}^tD_i')|_{V^M})$$

in the lemma. The uniqueness is clear from the surjectivity of $\Gamma_{0,x}^V$. \square

Let $\tilde{\mathcal{R}}$ be the subalgebra of \mathcal{R} generated by $\frac{1}{1-e^{-2\alpha}} \in \mathcal{M}$ ($\alpha \in \Sigma^+$). If we identify $\tilde{\mathcal{R}}$ with a subalgebra of $\text{End}_{\mathbb{C}} C^\infty(Wx)$, then it clearly has the following properties:

$$\begin{cases} w \tilde{\mathcal{R}} w^{-1} = \tilde{\mathcal{R}} & \text{for } w \in W, \\ [\partial(\xi), \tilde{\mathcal{R}}] \subset \tilde{\mathcal{R}} & \text{for } \xi \in \mathfrak{a}. \end{cases}$$

Hence it follows from Definition 4.2 that

$$\mathcal{T}(D) - \partial(D(\cdot - \rho)) \in (\tilde{\mathcal{R}} \cap \mathcal{M}) \partial(S(\mathfrak{a}_{\mathbb{C}})) W \quad \text{for } D \in S(\mathfrak{a}_{\mathbb{C}}).$$

This implies the next lemma, which can be considered as the Cherednik operator version of Lemma 5.4.

Lemma 5.6. *Suppose U is a W -module. For any $\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W$, the action of $\mathcal{T}(\Delta)$ on $C^\infty(Wx)$ induces its action on $\text{Hom}_W(U, C^\infty(Wx)) \simeq \text{Hom}_{\mathbb{C}}(U, C_x^\infty)$. On this action there exists a unique $F \in \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})) \otimes \text{End}_{\mathbb{C}} U^*$ such that*

$$\mathcal{T}(\Delta)\varphi = (\partial(\Delta(\cdot - \rho)) + F)\varphi \quad \text{for any } \varphi \in \text{Hom}_{\mathbb{C}}(U, C_x^\infty).$$

Suppose $V \in \hat{K}_M$ and $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K$. Let E_Δ be the E of Lemma 5.4 and

$$E_\Delta = \begin{pmatrix} E_{\text{single}} & P \\ Q & E_{\text{double}} \end{pmatrix}$$

its matrix expression by Remark 5.5 (iii). Moreover let $F_{\gamma(\Delta)}$ be the F of Lemma 5.6 for the W -module $U = V_{\text{single}}^M$ and $\gamma(\Delta) \in S(\mathfrak{a}_{\mathbb{C}})^W$. This can also be expressed in the matrix form with respect to the basis $\{v_1, \dots, v_{m'}\}$. By comparing Lemma 5.4 with Lemma 5.6 we can see: Theorem 5.1 (i) asserts that $E_{\text{single}} = F_{\gamma(\Delta)}$ and $P = 0$; Theorem 5.1 (ii) is equivalent to $Q = 0$. (Note that (iii) in the theorem is a corollary of (i).)

It is very interesting that if we confirm these things by some concrete calculations for one special case where $\Delta = L_{\mathfrak{g}}$ (the Casimir element of \mathfrak{g}), then all the other cases follow from it. Let us see this mechanism first.

Assume it is proved that we can take

$$E_{L_{\mathfrak{g}}} = \begin{pmatrix} F_{\gamma(L_{\mathfrak{g}})} & 0 \\ 0 & L_{\text{double}} \end{pmatrix}$$

as the E of Lemma 5.4 for $\Delta = L_{\mathfrak{g}}$. On the one hand, the commutativity

$$[r(L_{\mathfrak{g}}), r(\Delta)] = 0$$

implies

$$\left[\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + \begin{pmatrix} F_{\gamma(L_{\mathfrak{g}})} & 0 \\ 0 & L_{\text{double}} \end{pmatrix}, \partial(\gamma(\Delta)(\cdot - \rho)) + \begin{pmatrix} E_{\text{single}} & P \\ Q & E_{\text{double}} \end{pmatrix} \right] = 0.$$

This reduces to

$$(5.6) \quad [\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + F_{\gamma(L_{\mathfrak{g}})}, \partial(\gamma(\Delta)(\cdot - \rho)) + E_{\text{single}}] = 0,$$

$$(5.7) \quad (\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + F_{\gamma(L_{\mathfrak{g}})}) P - P (\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + L_{\text{double}}) = 0,$$

$$(5.8) \quad (\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + L_{\text{double}}) Q - Q (\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + F_{\gamma(L_{\mathfrak{g}})}) = 0, \\ [\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + L_{\text{double}}, \partial(\gamma(\Delta)(\cdot - \rho)) + E_{\text{double}}] = 0.$$

On the other hand, the commutativity

$$[\mathcal{T}(\gamma(L_{\mathfrak{g}})), \mathcal{T}(\gamma(\Delta))] = 0$$

implies

$$(5.9) \quad [\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + F_{\gamma(L_{\mathfrak{g}})}, \partial(\gamma(\Delta)(\cdot - \rho)) + F_{\gamma(\Delta)}] = 0.$$

From (5.6) and (5.9) we have

$$[\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), E_{\text{single}} - F_{\gamma(\Delta)}] = [E_{\text{single}} - F_{\gamma(\Delta)}, F_{\gamma(L_{\mathfrak{g}})}],$$

from (5.7)

$$[\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), P] = PL_{\text{double}} - F_{\gamma(L_{\mathfrak{g}})}P,$$

and from (5.8)

$$[\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), Q] = QF_{\gamma(L_{\mathfrak{g}})} - L_{\text{double}}Q.$$

Now applying the next lemma to these relations, we can get $E_{\text{single}} - F_{\gamma(\Delta)} = 0$, $P = 0$ and $Q = 0$.

Lemma 5.7. *Suppose matrices $S \in \text{Mat}(k, \ell; \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$, $T \in \text{Mat}(\ell, \ell; \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ and $U \in \text{Mat}(k, k; \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ satisfy*

$$(5.10) \quad [\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), S] = ST - US.$$

Then $S = 0$.

Proof. In general, any $S \in \text{Mat}(k, \ell; \mathcal{R} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ is uniquely expanded as

$$S = \sum_{\lambda \in \mathbb{Z}_{\geq 0} \Sigma^+} e^{\lambda} S_{\lambda} \quad \text{with } S_{\lambda} \in \text{Mat}(k, \ell; \partial(S(\mathfrak{a}_{\mathbb{C}})))$$

in the obvious way. Using this expansion we define

$$\text{spec } S = \{\lambda \in \mathbb{Z}_{\geq 0} \Sigma^+; S_{\lambda} \neq 0\}.$$

The condition $S \in \text{Mat}(k, \ell; \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ is equivalent to $0 \notin \text{spec } S$. This is also equivalent to

$$\text{spec}[\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), S] = \text{spec } S,$$

because a direct calculation shows

$$[\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), e^{\lambda}] = [\partial(L_{\mathfrak{a}} - 2H_{\rho}), e^{\lambda}] = e^{\lambda}(2\partial(H_{\lambda}) + B(H_{\lambda}, H_{\lambda} - 2H_{\rho}))$$

and this is non-zero unless $\lambda = 0$.

Now suppose S, T and U are as in the lemma and assume $S \neq 0$. Then there exists a minimal weight $\lambda_0 \neq 0$ in $\text{spec } S$ with respect to the partial order \preceq in the root lattice defined by

$$\lambda \preceq \mu \iff \mu - \lambda \in \mathbb{Z}_{\geq 0} \Sigma^+.$$

The above argument shows the ‘spec’ of the left-hand side of (5.10) must contain λ_0 . But it is easy to see that the ‘spec’ of the right-hand side of (5.10) does not contain λ_0 , a contradiction. \square

To make the above argument effective, we must prove Theorem 5.1 for $\Delta = L_{\mathfrak{g}}$. It is enough to show the following:

Proposition 5.8. *Let $x \in A_-$ and put $\bar{x} = xK \in G/K$ as before. Suppose $V \in \widehat{K}_M$ and $\Phi \in \text{Hom}_K(V, C^\infty(K\bar{x}))$. Then $r(L_{\mathfrak{g}})\Phi[v] = \ell(L_{\mathfrak{g}})\Phi[v]$. If $\gamma_{0,Wx}$ stands for the restriction map $C^\infty(K\bar{x}) \rightarrow C^\infty(Wx)$, then for any $v \in V_{\text{single}}^M$ it holds that*

$$\gamma_{0,Wx}(\ell(L_{\mathfrak{g}})\Phi[v]) = \mathcal{T}(\gamma(L_{\mathfrak{g}}))\gamma_{0,Wx}(\Phi[v]).$$

Moreover, if

$$\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(K\bar{x})) := \{\Phi \in \text{Hom}_K(V, C^\infty(K\bar{x})); \Gamma_{0,Wx}^V(\Phi)[V_{\text{double}}^M] = \{0\}\},$$

then $\ell(L_{\mathfrak{g}}) \circ \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(K\bar{x}))$.

Proof. The first assertion is clear since $L_{\mathfrak{g}}$ is a central element of $U(\mathfrak{g}_{\mathbb{C}})$ and ${}^t L_{\mathfrak{g}} = L_{\mathfrak{g}}$. Suppose $y = e^H$ ($H \in \mathfrak{a}$) is in a neighborhood of Wx . We may assume y is a regular point in A . Let $X_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Sigma$) and normalize it so that $-B(X_\alpha, \theta X_\alpha) = 1$. From $\text{Ad}(y)(X_\alpha + \theta X_\alpha) = \cosh \alpha(H)(X_\alpha + \theta X_\alpha) + \sinh \alpha(H)(X_\alpha - \theta X_\alpha)$ we have

$$X_\alpha - \theta X_\alpha \equiv -\coth \alpha(H)(X_\alpha + \theta X_\alpha) \pmod{\text{Ad}(y)(X_\alpha + \theta X_\alpha)U(\mathfrak{g}_{\mathbb{C}})}$$

and hence

$$\begin{aligned} (X_\alpha - \theta X_\alpha)^2 &\equiv -\coth \alpha(H)(X_\alpha + \theta X_\alpha)(X_\alpha - \theta X_\alpha) \\ &= -2\coth \alpha(H)H_\alpha - \coth \alpha(H)(X_\alpha - \theta X_\alpha)(X_\alpha + \theta X_\alpha) \\ &\equiv -2\coth \alpha(H)H_\alpha + \coth^2 \alpha(H)(X_\alpha + \theta X_\alpha)^2 \\ &\quad \pmod{\text{Ad}(y)(X_\alpha + \theta X_\alpha)U(\mathfrak{g}_{\mathbb{C}})}. \end{aligned}$$

Now suppose $v \in V^M$. Then

$$\begin{aligned} &\{\ell(X_\alpha + \theta X_\alpha)^2 - \ell(X_\alpha - \theta X_\alpha)^2\}\Phi[v](y) \\ &= \{2\coth \alpha(H)\ell(H_\alpha) + (1 - \coth^2 \alpha(H))\ell(X_\alpha + \theta X_\alpha)^2\}\Phi[v](y) \\ &= 2\coth \alpha(H)\ell(H_\alpha)\Phi[v](y) - \frac{1}{\sinh^2 \alpha(H)}\Phi[(X_\alpha + \theta X_\alpha)^2v](y) \\ &= -\coth \alpha(H)\partial(H_{2\alpha})\gamma_{0,Wx}(\Phi[v])(y) - \frac{1}{\sinh^2 \alpha(H)}\Phi[(X_\alpha + \theta X_\alpha)^2v](y). \end{aligned}$$

Let $L_{\mathfrak{a}} \in S(\mathfrak{a}_{\mathbb{C}})$ be as in Proposition 3.3. Let $\mathfrak{m} = \text{Lie } M$ and choose an orthonormal basis $\{Y_1, \dots, Y_{\dim \mathfrak{m}}\}$ of the Euclidean space $(\mathfrak{m}, -B(\cdot, \cdot))$. Put $L_{\mathfrak{m}} = -\sum_{i=1}^{\dim \mathfrak{m}} Y_i^2$.

Then $\ell(L_{\mathfrak{m}})\Phi[v] = \Phi[L_{\mathfrak{m}}v] = 0$. For each $\alpha \in \Sigma^+$ take a basis $\{X_\alpha^{(1)}, \dots, X_\alpha^{(\dim \mathfrak{g}_\alpha)}\}$ of \mathfrak{g}_α as in the proof of Lemma 3.6. Since

$$\begin{aligned} L_{\mathfrak{g}} &= L_{\mathfrak{m}} + L_{\mathfrak{a}} - \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{\dim \mathfrak{g}_\alpha} (X_\alpha^{(i)} \theta X_\alpha^{(i)} + \theta X_\alpha^{(i)} X_\alpha^{(i)}) \\ &= L_{\mathfrak{m}} + L_{\mathfrak{a}} - \frac{1}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{\dim \mathfrak{g}_\alpha} \left\{ (X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 - (X_\alpha^{(i)} - \theta X_\alpha^{(i)})^2 \right\}, \end{aligned}$$

we get

$$\begin{aligned} (5.11) \quad \ell(L_{\mathfrak{g}})\Phi[v](y) &= \left\{ \partial(L_{\mathfrak{a}}) + \sum_{\alpha \in \Sigma^+} \frac{\dim \mathfrak{g}_\alpha}{2} \coth \alpha(H) \partial(H_{2\alpha}) \right\} \gamma_{0, W_x}(\Phi[v])(y) \\ &\quad + \frac{1}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{\dim \mathfrak{g}_\alpha} \frac{1}{\sinh^2 \alpha(H)} \Phi[(X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 v](y). \end{aligned}$$

If $v \in V_{\text{single}}^M$, then from Lemma 2.7 (i) we have

$$\begin{aligned} \Phi[(X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 v](y) &= -|\alpha|^2 \Phi[(1 - s_\alpha)v](y) \\ &= -|\alpha|^2 (1 - s_\alpha) \gamma_{0, W_x}(\Phi[v])(y) \end{aligned}$$

and therefore

$$\begin{aligned} \ell(L_{\mathfrak{g}})\Phi[v](y) &= \left\{ \partial(L_{\mathfrak{a}}) + \sum_{\alpha \in \Sigma^+} \frac{\dim \mathfrak{g}_\alpha}{2} \left(\coth \alpha(H) \partial(H_{2\alpha}) - \frac{|\alpha|^2}{\sinh^2 \alpha(H)} (1 - s_\alpha) \right) \right\} \gamma_{0, W_x}(\Phi[v])(y) \\ &= \left\{ \partial(L_{\mathfrak{a}}) + \sum_{\beta \in R^+} \mathbf{m}(\beta) \left(\coth \frac{\beta(H)}{2} \partial(H_\beta) - \frac{|\beta|^2}{4 \sinh^2 \frac{\beta(H)}{2}} (1 - s_\beta) \right) \right\} \gamma_{0, W_x}(\Phi[v])(y) \\ &= \mathcal{T}(L_{\mathfrak{a}} - |\rho|^2) \gamma_{0, W_x}(\Phi[v])(y) \quad (\because (4.2)) \\ &= \mathcal{T}(\gamma(L_{\mathfrak{g}})) \gamma_{0, W_x}(\Phi[v])(y). \quad (\because \gamma(L_{\mathfrak{g}}) = L_{\mathfrak{a}} - |\rho|^2) \end{aligned}$$

This proves the second assertion. Finally, if $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(K\bar{x}))$ and $v \in V_{\text{double}}^M$, then it follows from Theorem 2.3 (ii) and Lemma 2.7 (ii) that

$$\Phi[(X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 v](y) = \Phi[p^V((X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 v)](y) = 0.$$

Thus, in this case, the right-hand side of (5.11) vanishes. \square

6. HARISH-CHANDRA HOMOMORPHISMS

The radial part formula given in the last section concerns the action of $r(U(\mathfrak{g}_{\mathbb{C}})^K)$ on $C^\infty(G/K)$. Its formulation is relatively simple since operators are always K -invariant. In the next section we shall develop another kind of radial part formula concerning the action of $\ell(U(\mathfrak{g}_{\mathbb{C}}))$ on $C^\infty(G/K)$, in which we treat the case where both operators

and functions are non- K -invariant. To do so, we must first prepare a non- K -invariant generalization of the Harish-Chandra homomorphism.

In general, for a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module \mathcal{Y} we put $\Gamma(\mathcal{Y}) = (\mathcal{Y}/\mathfrak{n}_{\mathbb{C}}\mathcal{Y})^M$. Since the 0th $\mathfrak{n}_{\mathbb{C}}$ -homology $\mathcal{Y}/\mathfrak{n}_{\mathbb{C}}\mathcal{Y}$ of \mathcal{Y} is $(\mathfrak{m}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}}, M)$ -module, its M -fixed part $\Gamma(\mathcal{Y})$ is naturally an $\mathfrak{a}_{\mathbb{C}}$ -module. But in various contexts it is useful to endow $\Gamma(\mathcal{Y})$ with a shifted (dotted) $\mathfrak{a}_{\mathbb{C}}$ -module structure. That is, we let $\xi \in \mathfrak{a}_{\mathbb{C}}$ act on $y \in \Gamma(\mathcal{Y})$ by $\xi \cdot y = (\xi - \rho(\xi))y$. Note there is a natural linear surjective map

$$(6.1) \quad \gamma^{\mathcal{Y}} : \mathcal{Y} \twoheadrightarrow \mathcal{Y}/\mathfrak{n}_{\mathbb{C}}\mathcal{Y} \twoheadrightarrow \Gamma(\mathcal{Y}) = (\mathcal{Y}/\mathfrak{n}_{\mathbb{C}}\mathcal{Y})^M$$

where the second map in this composition is the projection to the isotypic component of the trivial representation of M . If there is no fear of confusion, we use a brief symbol γ for $\gamma^{\mathcal{Y}}$ because when $\mathcal{Y} = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathbb{C}_{\text{triv}}$ this map essentially coincides with γ of (1.1) (Example 6.1).

For any $V \in \widehat{K}_M$ we define a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module $P_G(V) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V$. If $(V^M)^{\perp}$ denotes the orthogonal compliment of V^M with respect to a K -invariant inner product of V , then we have the direct sum decomposition

$$(6.2) \quad P_G(V) = (\mathfrak{n}_{\mathbb{C}}U(\mathfrak{g}_{\mathbb{C}}) \otimes V \oplus S(\mathfrak{a}_{\mathbb{C}}) \otimes (V^M)^{\perp}) \oplus S(\mathfrak{a}_{\mathbb{C}}) \otimes V^M.$$

Hence $\Gamma(P_G(V)) \simeq S(\mathfrak{a}_{\mathbb{C}}) \otimes V^M$. On the other hand, for any W -module U we put $P_{\mathbf{H}}(U) := \mathbf{H} \otimes_{\mathbb{C}W} U$. This is an \mathbf{H} -module and naturally $P_{\mathbf{H}}(U) \simeq S(\mathfrak{a}_{\mathbb{C}}) \otimes U$ as an $\mathfrak{a}_{\mathbb{C}}$ -module. Now let us identify $\Gamma(P_G(V))$ with $P_{\mathbf{H}}(V^M)$ by

$$(6.3) \quad \Gamma(P_G(V)) \simeq S(\mathfrak{a}_{\mathbb{C}}) \otimes V^M \ni \varphi(\lambda) \otimes v \mapsto \varphi(\lambda + \rho) \otimes v \in S(\mathfrak{a}_{\mathbb{C}}) \otimes V^M \simeq P_{\mathbf{H}}(V^M).$$

Note this is an isomorphism of $\mathfrak{a}_{\mathbb{C}}$ -modules.

Example 6.1. If $V = \mathbb{C}_{\text{triv}}$ then for any fixed $v_{\text{triv}} \in \mathbb{C}_{\text{triv}} \setminus \{0\}$ we have the surjection $U(\mathfrak{g}_{\mathbb{C}}) \ni D \mapsto D \otimes v_{\text{triv}} \in P_G(\mathbb{C}_{\text{triv}})$ and the bijection $S(\mathfrak{a}_{\mathbb{C}}) \ni \varphi \mapsto \varphi \otimes v_{\text{triv}} \in P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})$, for which the diagram

$$\begin{array}{ccc} U(\mathfrak{g}_{\mathbb{C}}) & \twoheadrightarrow & P_G(\mathbb{C}_{\text{triv}}) \\ \gamma \text{ in (1.1)} \downarrow & & \downarrow \gamma = \gamma^{P_G(\mathbb{C}_{\text{triv}})} \\ S(\mathfrak{a}_{\mathbb{C}}) & \xrightarrow{\sim} & P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}) \end{array}$$

commutes.

Now suppose \mathcal{Y}_1 and \mathcal{Y}_2 are two $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules and $\Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(\mathcal{Y}_1, \mathcal{Y}_2)$. Then there exists a unique $\Gamma(\Psi) \in \text{Hom}_{\mathfrak{a}_{\mathbb{C}}}(\Gamma(\mathcal{Y}_1), \Gamma(\mathcal{Y}_2))$ such that the diagram

$$(6.4) \quad \begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{\Psi} & \mathcal{Y}_2 \\ \gamma \downarrow & & \downarrow \gamma \\ \Gamma(\mathcal{Y}_1) & \xrightarrow{\Gamma(\Psi)} & \Gamma(\mathcal{Y}_2) \end{array}$$

commutes. Thus Γ defines a (right exact) functor from the category of $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules to the category of $\mathfrak{a}_{\mathbb{C}}$ -modules.

Definition 6.2. Suppose $E, V \in \widehat{K}_M$. Then naturally

$$P_{\mathbf{H}}(E^M) = P_{\mathbf{H}}(E_{\text{single}}^M) \oplus P_{\mathbf{H}}(E_{\text{double}}^M), \quad P_{\mathbf{H}}(V^M) = P_{\mathbf{H}}(V_{\text{single}}^M) \oplus P_{\mathbf{H}}(V_{\text{double}}^M).$$

For $\Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V))$ we define $\tilde{\Gamma}(\Psi) \in \text{Hom}_{\mathfrak{a}_{\mathbb{C}}}(P_{\mathbf{H}}(E_{\text{single}}^M), P_{\mathbf{H}}(V_{\text{single}}^M))$ by

$$P_{\mathbf{H}}(E_{\text{single}}^M) \hookrightarrow P_{\mathbf{H}}(E^M) \xrightarrow{\Gamma(\Psi)} P_{\mathbf{H}}(V^M) \twoheadrightarrow P_{\mathbf{H}}(V_{\text{single}}^M).$$

In addition, we put

$$\begin{aligned} \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{1 \rightarrow 1}(P_G(E), P_G(V)) \\ &= \{ \Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V)); \Gamma(\Psi)[P_{\mathbf{H}}(E_{\text{single}}^M)] \subset P_{\mathbf{H}}(V_{\text{single}}^M) \}, \\ \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(E), P_G(V)) \\ &= \{ \Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V)); \Gamma(\Psi)[P_{\mathbf{H}}(E_{\text{double}}^M)] \subset P_{\mathbf{H}}(V_{\text{double}}^M) \}. \end{aligned}$$

In general, the correspondence $\tilde{\Gamma}$ does not commute with composition of morphisms. But it does in the following cases:

Proposition 6.3. Suppose $E, F, V \in \widehat{K}_M$, $\Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(F))$ and $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(F), P_G(V))$. If $\Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{1 \rightarrow 1}$ or $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}$ then $\tilde{\Gamma}(\Phi \circ \Psi) = \tilde{\Gamma}(\Phi) \circ \tilde{\Gamma}(\Psi)$.

Remark 6.4. Suppose $E, V \in \widehat{K}_M$.

(i) If $V \in \widehat{K}_{\text{sp}}$ or $E \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$, then

$$\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{1 \rightarrow 1}(P_G(E), P_G(V)) = \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V)).$$

If $E \in \widehat{K}_{\text{sp}}$ or $V \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$, then

$$\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(E), P_G(V)) = \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V)).$$

(ii) If $V \in \widehat{K}_{\text{sp}}$, then

$$\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(E), P_G(V)) = \{ \Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V)); \Gamma(\Psi)[P_{\mathbf{H}}(E_{\text{double}}^M)] = \{0\} \}.$$

We are now in the position to state the main result of this section.

Theorem 6.5 (the generalized Harish-Chandra homomorphism). Suppose $E, V \in \widehat{K}_M$.

(i) $\tilde{\Gamma}(\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V))) \subset \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(E_{\text{single}}^M), P_{\mathbf{H}}(V_{\text{single}}^M))$.

(ii) If $V = \mathbb{C}_{\text{triv}}$ then $\tilde{\Gamma}$ induces a bijection

$$\tilde{\Gamma} : \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(E), P_G(\mathbb{C}_{\text{triv}})) \xrightarrow{\sim} \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(E_{\text{single}}^M), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})).$$

(iii) If $E = \mathbb{C}_{\text{triv}}$ then $\tilde{\Gamma}$ induces a bijection

$$\tilde{\Gamma} : \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{1 \rightarrow 1}(P_G(\mathbb{C}_{\text{triv}}), P_G(V)) \xrightarrow{\sim} \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(V_{\text{single}}^M)).$$

(iv) If $E = V = \mathbb{C}_{\text{triv}}$ then there exist natural identifications $\text{End}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(\mathbb{C}_{\text{triv}})) \simeq U(\mathfrak{g}_{\mathbb{C}})^K / (U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}})^K$ and $\text{End}_{\mathbf{H}}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})) \simeq S(\mathfrak{a}_{\mathbb{C}})^W$, under which the algebra isomorphism

$$\tilde{\Gamma} : \text{End}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(\mathbb{C}_{\text{triv}})) \xrightarrow{\sim} \text{End}_{\mathbf{H}}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$$

coincides with (1.7).

Before proving the theorem we introduce some notation, which will also be used to formulate the radial part formula in the next section.

Definition 6.6. Suppose $E, V \in \widehat{K}_M$. By the Frobenius reciprocity, we can identify

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V)) &\simeq \mathrm{Hom}_K(E, P_G(V)), \\ \mathrm{Hom}_{\mathfrak{a}_{\mathbb{C}}}(P_{\mathbf{H}}(E^M), P_{\mathbf{H}}(V^M)) &\simeq \mathrm{Hom}_{\mathbb{C}}(E^M, P_{\mathbf{H}}(V^M)), \\ \mathrm{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(E_{\mathrm{single}}^M), P_{\mathbf{H}}(V_{\mathrm{single}}^M)) &\simeq \mathrm{Hom}_W(E_{\mathrm{single}}^M, P_{\mathbf{H}}(V_{\mathrm{single}}^M)). \end{aligned}$$

Under these identifications, the map

$$\Gamma : \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V)) \rightarrow \mathrm{Hom}_{\mathfrak{a}_{\mathbb{C}}}(P_{\mathbf{H}}(E^M), P_{\mathbf{H}}(V^M))$$

reduces to

$$\begin{aligned} \Gamma_V^E : \mathrm{Hom}_K(E, P_G(V)) &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(E^M, P_{\mathbf{H}}(V^M)); \\ \Psi &\longmapsto (\psi : E^M \hookrightarrow E \xrightarrow{\Psi} P_G(V) \xrightarrow{\gamma} P_{\mathbf{H}}(V^M)). \end{aligned}$$

This interpretation is distinguished by attaching super- and sub-scripts to Γ . We define

$$\tilde{\Gamma}_V^E : \mathrm{Hom}_K(E, P_G(V)) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(E_{\mathrm{single}}^M, P_{\mathbf{H}}(V_{\mathrm{single}}^M))$$

similarly (Theorem 6.5 (i) asserts that the target space of this map can be replaced with Hom_W). Finally we put

$$\begin{aligned} \mathrm{Hom}_K^{1 \rightarrow 1}(E, P_G(V)) &= \{\Psi \in \mathrm{Hom}_K(E, P_G(V)); \Gamma_V^E(\Psi)[E_{\mathrm{single}}^M] \subset P_{\mathbf{H}}(V_{\mathrm{single}}^M)\}, \\ \mathrm{Hom}_K^{2 \rightarrow 2}(E, P_G(V)) &= \{\Psi \in \mathrm{Hom}_K(E, P_G(V)); \Gamma_V^E(\Psi)[E_{\mathrm{double}}^M] \subset P_{\mathbf{H}}(V_{\mathrm{double}}^M)\}. \end{aligned}$$

Proof of Theorem 6.5. From [O, Theorem 4.7] it holds that

$$\tilde{\Gamma}_{\mathbb{C}_{\mathrm{triv}}}^E(\mathrm{Hom}_K(E, P_G(\mathbb{C}_{\mathrm{triv}}))) \subset \mathrm{Hom}_W(E_{\mathrm{single}}^M, P_{\mathbf{H}}(\mathbb{C}_{\mathrm{triv}})),$$

which is equivalent to (i) for $V = \mathbb{C}_{\mathrm{triv}}$. Also, by [O, Theorem 4.11] we have an isomorphism

$$\tilde{\Gamma}_{\mathbb{C}_{\mathrm{triv}}}^E : \mathrm{Hom}_K^{2 \rightarrow 2}(E, P_G(\mathbb{C}_{\mathrm{triv}})) \xrightarrow{\sim} \mathrm{Hom}_W(E_{\mathrm{single}}^M, P_{\mathbf{H}}(\mathbb{C}_{\mathrm{triv}})),$$

which is equivalent to (ii).

Next, fix a non-zero $v_{\mathrm{triv}} \in \mathbb{C}_{\mathrm{triv}}$. Then (iv) is clear from

$$\begin{aligned} \mathrm{End}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(\mathbb{C}_{\mathrm{triv}})) &= \mathrm{Hom}_K(\mathbb{C}_{\mathrm{triv}}, P_G(\mathbb{C}_{\mathrm{triv}})) \\ &= \mathrm{Hom}_{\mathbb{C}}(\mathbb{C} v_{\mathrm{triv}}, U(\mathfrak{g}_{\mathbb{C}})^K / (U(\mathfrak{g}_{\mathbb{C}}) \mathfrak{k}_{\mathbb{C}})^K \otimes v_{\mathrm{triv}}), \\ \mathrm{End}_{\mathbf{H}}(P_{\mathbf{H}}(\mathbb{C}_{\mathrm{triv}})) &= \mathrm{Hom}_W(\mathbb{C}_{\mathrm{triv}}, P_{\mathbf{H}}(\mathbb{C}_{\mathrm{triv}})) \\ &= \mathrm{Hom}_{\mathbb{C}}(\mathbb{C} v_{\mathrm{triv}}, S(\mathfrak{a}_{\mathbb{C}})^W \otimes v_{\mathrm{triv}}) \end{aligned}$$

and Example 6.1.

To prove (i) suppose $E, V \in \widehat{K}_M$ are arbitrary. Choose a basis $\{v_1, \dots, v_{m'}\}$ of V_{single}^M . Let $H_W(\mathfrak{a}_{\mathbb{C}}) \subset S(\mathfrak{a}_{\mathbb{C}})$ be the space of W -harmonic polynomials on \mathfrak{a}^* . Then there exist m' ($= \dim V_{\mathrm{single}}^M$) linearly independent W -homomorphisms $\varphi_j : V_{\mathrm{single}}^M \rightarrow H_W(\mathfrak{a}_{\mathbb{C}})$ ($j = 1, \dots, m'$) such that $\varphi_j[v_1], \dots, \varphi_j[v_{m'}]$ are all homogeneous with the same degree for each fixed j . For $j = 1, \dots, m'$ choose $\hat{\varphi}_j \in \mathrm{Hom}_W(V_{\mathrm{single}}^M, P_{\mathbf{H}}(\mathbb{C}_{\mathrm{triv}}))$ so that the top

degree part of $\hat{\varphi}_j$ coincides with φ_j . Here we are identifying $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}) = S(\mathfrak{a}_{\mathbb{C}}) \otimes v_{\text{triv}}$ with $S(\mathfrak{a}_{\mathbb{C}})$ naturally. Note that this identification respects $\mathfrak{a}_{\mathbb{C}}$ -module structures and that $\det(\hat{\varphi}_j[v_i])_{1 \leq i, j \leq m'} \neq 0$ since $\det(\varphi_j[v_i])_{1 \leq i, j \leq m'} \neq 0$ (cf. [HC, §2]). Thus the \mathbf{H} -homomorphism

$$P_{\mathbf{H}}(V_{\text{single}}^M) \xrightarrow{\prod_j \hat{\varphi}_j} P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})^{m'};$$

$$\sum_i f_i \otimes v_i \mapsto \left(\sum_i f_i \hat{\varphi}_1[v_i] \otimes v_{\text{triv}}, \dots, \sum_i f_i \hat{\varphi}_{m'}[v_i] \otimes v_{\text{triv}}, \right)$$

is injective. Now using (ii) we can lift $\hat{\varphi}_j$ to $\Phi_j \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(V), P_G(\mathbb{C}_{\text{triv}}))$. By virtue of Proposition 6.3, for any $\Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V))$ we have

$$\prod_j \tilde{\Gamma}(\Phi_j \circ \Psi) = \prod_j (\tilde{\Gamma}(\Phi_j) \circ \tilde{\Gamma}(\Psi)) = \prod_j (\hat{\varphi}_j \circ \tilde{\Gamma}(\Psi)) = \left(\prod_j \hat{\varphi}_j \right) \circ \tilde{\Gamma}(\Psi).$$

The leftmost side shows this is an \mathbf{H} -homomorphism of $P_{\mathbf{H}}(E_{\text{single}}^M)$ into $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})^{m'}$ since we already know (i) is valid for $\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(\mathbb{C}_{\text{triv}}))$. But since $\prod_j \hat{\varphi}_j$ is an injective \mathbf{H} -homomorphism, the rightmost side shows $\tilde{\Gamma}(\Psi)$ is also an \mathbf{H} -homomorphism.

We postpone the proof of (iii) until we introduce the notion of *star operations* for morphisms in §10. With that notion, (iii) is equivalent to (ii) (Corollary 10.13). \square

7. RADIAL PART FORMULA, II

In this section we try to generalize (1.4) to some cases where Δ and f are not necessarily K -invariant. In view of (1.4) the radial part of a left Lie algebra action on the K -invariant functions is twisted by the Cartan involution θ . Related to this, we introduce the “Cartan involution” for \mathbf{H} .

Definition 7.1. Let w_0 be the longest element of W . We define the algebra automorphism $\theta_{\mathbf{H}}$ of \mathbf{H} so that it satisfies the following relations:

$$\begin{cases} \theta_{\mathbf{H}} w = w & \text{for } w \in W, \\ \theta_{\mathbf{H}} \xi = -w_0 w_0(\xi) w_0 & \text{for } \xi \in \mathfrak{a}_{\mathbb{C}}. \end{cases}$$

The automorphism is well defined by (4.1).

For $V \in \hat{K}_M$, θ naturally induces a K -linear automorphism of $P_G(V) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{k}_{\mathbb{C}}} V$:

$$P_G(V) \ni D \otimes v \mapsto (\theta D) \otimes v \in P_G(V).$$

For a W -module U , $\theta_{\mathbf{H}}$ naturally induces a W -linear automorphism of $P_{\mathbf{H}}(U)$ likewise.

Proposition 7.2. Let $\bar{w}_0 \in N_K(\mathfrak{a}) = \{k \in K; \text{Ad}(k)(\mathfrak{a}) \subset \mathfrak{a}\}$ be an element normalizing \mathfrak{a} with the same action as w_0 . Suppose $V \in \hat{K}_M$. Then it holds that

$$(7.1) \quad \gamma(\theta(\bar{w}_0 D)) = \theta_{\mathbf{H}}(w_0 \gamma(D)) \quad \text{for any } D \in P_G(V).$$

Moreover, if $E, V \in \hat{K}_M$ then we have

$$(7.2) \quad \tilde{\Gamma}_V^E(\theta \circ \Psi) = \theta_{\mathbf{H}} \circ \tilde{\Gamma}_V^E(\Psi) \quad \text{for any } \Psi \in \text{Hom}_K(E, P_G(V)),$$

$$(7.3) \quad \theta \circ \Psi \in \text{Hom}_K^{1 \rightarrow 1}(E, P_G(V)) \quad \text{for any } \Psi \in \text{Hom}_K^{1 \rightarrow 1}(E, P_G(V)),$$

$$(7.4) \quad \theta \circ \Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, P_G(V)) \quad \text{for any } \Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, P_G(V)).$$

Proof. Suppose $V \in \widehat{K}_M$ and $D \in P_G(V)$. Let $D = D_1 + D_2$ be the decomposition corresponding to the direct sum decomposition (6.2). Applying $\theta \circ \bar{w}_0$ to this, we get $\theta(\bar{w}_0 D) = \theta(\bar{w}_0 D_1) + \theta(\bar{w}_0 D_2)$, which is nothing but the decomposition of $\theta(\bar{w}_0 D)$ corresponding to (6.2). Thus if we write the identification (6.3) in the form

$$\iota : S(\mathfrak{a}_{\mathbb{C}}) \otimes V^M \ni \varphi(\lambda) \otimes v \longmapsto \varphi(\lambda + \rho) \otimes v \in S(\mathfrak{a}_{\mathbb{C}}) \otimes V^M \simeq P_{\mathbf{H}}(V^M)$$

and prove the equality $\iota(\theta(\bar{w}_0 D_2)) = \theta_{\mathbf{H}}(w_0 \iota(D_2))$, then (7.1) follows. But the equality holds since

$$\begin{aligned} \iota(\theta(\bar{w}_0 \varphi(\lambda) \otimes v)) &= \iota(\varphi(-w_0^{-1} \lambda) \otimes w_0 v) = \iota(\varphi(-w_0 \lambda) \otimes w_0 v) \\ &= \varphi(-w_0(\lambda + \rho)) \otimes w_0 v = \varphi(-w_0 \lambda + \rho) \otimes w_0 v \\ &= w_0 w_0 \varphi(-w_0 \lambda + \rho) w_0 \otimes v = w_0 \theta_{\mathbf{H}}(\varphi(\lambda + \rho) \otimes v) \\ &= w_0 \theta_{\mathbf{H}}(\iota(\varphi(\lambda) \otimes v)) = \theta_{\mathbf{H}}(w_0 \iota(\varphi(\lambda) \otimes v)). \end{aligned}$$

Now, suppose $E, V \in \widehat{K}_M$ and $\Psi \in \text{Hom}_K(E, P_G(V))$. For any $e \in E^M$

$$\begin{aligned} \Gamma_V^E(\theta \circ \Psi)[e] &= \gamma(\theta(\Psi[e])) = \gamma(\theta(\Psi[w_0 w_0 e])) \\ &= \gamma(\theta(\bar{w}_0 \Psi[w_0 e])) & (\because \Psi \in \text{Hom}_K) \\ &= \theta_{\mathbf{H}}(w_0 \gamma(\Psi[w_0 e])) & (\because (7.1)) \\ &= \theta_{\mathbf{H}}(w_0 \Gamma_V^E(\Psi)[w_0 e]). \end{aligned}$$

This expression proves (7.3) since $w_0 E_{\text{single}}^M = E_{\text{single}}^M$ and both $\theta_{\mathbf{H}}$ and the left multiplication by w_0 leave $P_{\mathbf{H}}(V_{\text{single}}^M)$ stable. Similar is (7.4). Lastly, since the projection $P_{\mathbf{H}}(V^M) \rightarrow P_{\mathbf{H}}(V_{\text{single}}^M)$ commutes with $\theta_{\mathbf{H}}$ and the left multiplication by w_0 , for $e \in E_{\text{single}}^M$ we have

$$\begin{aligned} \tilde{\Gamma}_V^E(\theta \circ \Psi)[e] &= \theta_{\mathbf{H}}(w_0 \tilde{\Gamma}_V^E(\Psi)[w_0 e]) \\ &= \theta_{\mathbf{H}}(w_0 w_0 \tilde{\Gamma}_V^E(\Psi)[e]) = \theta_{\mathbf{H}}(\tilde{\Gamma}_V^E(\Psi)[e]) & (\because \text{Theorem 6.5 (i)}) \\ &= (\theta_{\mathbf{H}} \circ \tilde{\Gamma}_V^E(\Psi))[e]. \end{aligned}$$

This shows (7.2). □

Suppose $V \in \widehat{K}_M$. As in Definition 6.6 we identify

$$\text{Hom}_K(V, C^\infty(G/K)) \simeq \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), C^\infty(G/K)_{K\text{-finite}})$$

using the action of $\ell(U(\mathfrak{g}_{\mathbb{C}}))$ on $C^\infty(G/K)_{K\text{-finite}}$. In view of (1.4) and Proposition 7.2 we let the analogous identification

$$\text{Hom}_W(U, C^\infty(A)) \simeq \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(U), C^\infty(A))$$

for any W -module U be based on the \mathbf{H} -module structure of $C^\infty(A)$ defined by $\mathcal{T}(\theta_{\mathbf{H}} \cdot)$. Under these identifications the map Γ_0^V defined by (2.2) can be rewritten as

$$\begin{aligned} \Gamma_0 : \text{Hom}_{\mathfrak{g}_{\mathbf{C}}, K}(P_G(V), C^\infty(G/K)_{K\text{-finite}}) &\rightarrow \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V^M), C^\infty(A)); \\ \Phi &\longmapsto \left(\varphi : P_{\mathbf{H}}(V^M) \ni \sum_{i=1}^m h_i \otimes v_i \longmapsto \sum_{i=1}^m \mathcal{T}(\theta_{\mathbf{H}} h_i) \gamma_0(\Phi[v_i]) \right). \end{aligned}$$

We distinguish this interpretation by the symbol Γ_0 with no superscript. We remark in contrast to (6.4) the diagram

$$\begin{array}{ccc} P_G(V) & \xrightarrow{\Phi} & C^\infty(G/K)_{K\text{-finite}} \\ \gamma \downarrow & & \downarrow \gamma_0 \\ P_{\mathbf{H}}(V^M) & \xrightarrow{\Gamma_0(\Phi)} & C^\infty(A) \end{array}$$

cannot be assumed commutative at all. Similarly, for $\Phi \in \text{Hom}_K(V, C^\infty(G/K)) = \text{Hom}_{\mathfrak{g}_{\mathbf{C}}, K}(P_G(V), C^\infty(G/K)_{K\text{-finite}})$ we let $\tilde{\Gamma}_0(\Phi) \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), C^\infty(A))$ be a map identified with $\tilde{\Gamma}_0^V(\Phi) \in \text{Hom}_W(V_{\text{single}}^M, C^\infty(A))$ (cf. Definition 2.5 (ii)). We also use the following identification:

$$\begin{aligned} &\text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(G/K)) \\ &\simeq \text{Hom}_{\mathfrak{g}_{\mathbf{C}}, K}^{2 \rightarrow 2}(P_G(V), C^\infty(G/K)_{K\text{-finite}}) \\ &:= \{ \Phi \in \text{Hom}_{\mathfrak{g}_{\mathbf{C}}, K}(P_G(V), C^\infty(G/K)_{K\text{-finite}}); \Gamma_0(\Phi)[P_{\mathbf{H}}(V_{\text{double}}^M)] = \{0\} \}. \end{aligned}$$

Theorem 7.3 (the radial part formula). *Suppose $E, V \in \hat{K}_M$, $\Psi \in \text{Hom}_K(E, P_G(V))$ and $\Phi \in \text{Hom}_K(V, C^\infty(G/K)) = \text{Hom}_{\mathfrak{g}_{\mathbf{C}}, K}(P_G(V), C^\infty(G/K)_{K\text{-finite}})$.*

(i) *Suppose $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(G/K)) = \text{Hom}_{\mathfrak{g}_{\mathbf{C}}, K}^{2 \rightarrow 2}(P_G(V), C^\infty(G/K)_{K\text{-finite}})$. Then it holds that*

$$\tilde{\Gamma}_0^E(\Phi \circ \Psi) = \tilde{\Gamma}_0(\Phi) \circ \tilde{\Gamma}_V^E(\Psi).$$

This means the diagram

$$\begin{array}{ccccc} E & \xrightarrow{\Psi} & P_G(V) & \xrightarrow{\Phi} & C^\infty(G/K)_{K\text{-finite}} \\ \uparrow & & & & \downarrow \gamma_0 \\ E_{\text{single}}^M & \xrightarrow{\tilde{\Gamma}_V^E(\Psi)} & P_{\mathbf{H}}(V_{\text{single}}^M) & \xrightarrow{\tilde{\Gamma}_0(\Phi)} & C^\infty(A) \end{array}$$

is commutative. In other words, if we take a basis $\{v_1, \dots, v_n\}$ of V so that $\{v_1, \dots, v_{m'}\}$, $\{v_{m'+1}, \dots, v_m\}$ and $\{v_{m+1}, \dots, v_n\}$ are bases of V_{single}^M , V_{double}^M and $(V^M)^\perp$ respectively ($m' \leq m \leq n$), and if for any $e \in E_{\text{single}}^M$ we write

$$\Psi[e] = \sum_{i=1}^n D_i \otimes v_i \text{ with } D_i \in U(\mathfrak{g}_{\mathbf{C}}), \quad \Gamma_V^E(\Psi)[e] = \sum_{i=1}^m h_i \otimes v_i \text{ with } h_i \in \mathbf{H},$$

then

$$\gamma_0 \left(\sum_{i=1}^n \ell(D_i) \Phi[v_i] \right) = \sum_{i=1}^{m'} \mathcal{T}(\theta_{\mathbf{H}} h_i) \gamma_0(\Phi[v_i]).$$

(ii) If $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), C^\infty(G/K)_{K\text{-finite}})$ and $\Psi \in \text{Hom}_K^{1 \rightarrow 1}(E, P_G(V))$ (see Definition 6.6), then the same assertion as (i) holds.

(iii) If $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(V), C^\infty(G/K)_{K\text{-finite}})$ and $\Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, P_G(V))$ (see Definition 6.6), then $\Phi \circ \Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, C^\infty(G/K))$ and hence it holds that

$$\Gamma_0^E(\Phi \circ \Psi) = \Gamma_0(\Phi) \circ \Gamma_V^E(\Psi).$$

The proof of the theorem is similar to that of Theorem 5.1. Suppose $x \in A_-$ and let $\gamma_{0, Wx} : C^\infty(K\bar{x}) \rightarrow C^\infty(Wx)$, $\gamma_{0, x} : C^\infty(K\bar{x}) \rightarrow C_x^\infty$, \mathcal{R} , \mathcal{M} and $\mathcal{R} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}}))$ be as in §5. In addition to the basis $\{v_1, \dots, v_n\}$ of V in (i), take a basis $\{e_1, \dots, e_\mu\}$ of E^M so that $\{e_1, \dots, e_{\mu'}\}$, $\{e_{\mu'+1}, \dots, e_\mu\}$ are bases of E_{single}^M , E_{double}^M respectively ($\mu' \leq \mu$). We use the same identifications with (5.4):

$$\text{Hom}_W(V^M, C^\infty(Wx)) \simeq \text{Hom}_{\mathbb{C}}(V^M, C_x^\infty), \quad \text{Hom}_W(E^M, C^\infty(Wx)) \simeq \text{Hom}_{\mathbb{C}}(E^M, C_x^\infty).$$

They are still identified with the spaces of column vectors with entries in C_x^∞ by using $\{v_1, \dots, v_m\}$ and $\{e_1, \dots, e_\mu\}$. Thus each element of $\text{Mat}(\mu, m; \mathcal{R} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ naturally defines a map $\text{Hom}_W(V^M, C^\infty(Wx)) \rightarrow \text{Hom}_W(E^M, C^\infty(Wx))$.

Lemma 7.4. Suppose $\Psi \in \text{Hom}_K(E, P_G(V))$ and take $D_{ij} \in U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})$ so that

$$(7.5) \quad (\theta \circ \Psi)[e_i] = \sum_{j=1}^n D_{ij} \otimes v_j \quad \text{for } i = 1, \dots, \mu.$$

Then there exists a unique $S = (S_{ij}) \in \text{Mat}(\mu, m; \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ such that

$$(7.6) \quad \Gamma_{0, x}^E(\Phi \circ \Psi)[e_i] = \sum_{j=1}^m \left(\partial(\gamma(D_{ij})(\cdot - \rho)) + S_{ij} \right) \Gamma_{0, x}^V(\Phi)[v_j]$$

for any $i = 1, \dots, \mu$ and $\Phi \in \text{Hom}_K(V, C^\infty(K\bar{x})) \simeq \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), C^\infty(K\bar{x})_{K\text{-finite}})$.

Proof. By the same method used in obtaining (5.5) we can take $c_{ijk} \in \mathcal{M}$, $D'_{ijk} \in U(\mathfrak{k}_{\mathbb{C}})$, and $D''_{ijk} \in S(\mathfrak{a}_{\mathbb{C}})$ ($1 \leq i \leq \mu, 1 \leq j \leq n, 1 \leq k \leq q_{ij}$) so that it holds that

$$(\theta \circ \Psi)[e_i] = \sum_{j=1}^n \gamma(D_{ij})(\cdot - \rho) \otimes v_j + \sum_{j=1}^n \sum_{k=1}^{q_{ij}} c_{ijk}(y) \text{Ad}(y^{-1})(D'_{ijk}) D''_{ijk} \otimes v_j$$

for any $i = 1, \dots, \mu$ and $y \in A_-$. Applying θ to both the side we get

$$\Psi[e_i] = \sum_{j=1}^n {}^t(\gamma(D_{ij})(\cdot - \rho)) \otimes v_j + \sum_{j=1}^n \sum_{k=1}^{q_{ij}} c_{ijk}(y) \text{Ad}(y)(D'_{ijk}) {}^t D''_{ijk} \otimes v_j.$$

Hence letting $\epsilon : U(\mathfrak{k}_{\mathbb{C}}) \rightarrow \mathbb{C}$ be the projection of the decomposition $U(\mathfrak{k}_{\mathbb{C}}) = \mathbb{C} \oplus U(\mathfrak{k}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}$ to the first summand, we calculate for $i = 1, \dots, \mu$ and y in a neighborhood of x

$$(\Phi \circ \Psi)[e_i](y)$$

$$\begin{aligned}
&= \sum_{j=1}^n \ell\left({}^t(\gamma(D_{ij})(\cdot - \rho))\right) \Phi[v_j](y) + \sum_{j=1}^n \sum_{k=1}^{q_{ij}} c_{ijk}(y) \ell(\text{Ad}(y)(D'_{ijk}) {}^t D''_{ijk}) \Phi[v_j](y) \\
&= \sum_{j=1}^n \partial(\gamma(D_{ij})(\cdot - \rho)) \gamma_{0,x}(\Phi[v_j])(y) + \sum_{j=1}^n \sum_{k=1}^{q_{ij}} c_{ijk}(y) \epsilon(D'_{ijk}) \partial(D''_{ijk}) \gamma_{0,x}(\Phi[v_j])(y) \\
&= \sum_{j=1}^m \partial(\gamma(D_{ij})(\cdot - \rho)) \gamma_{0,x}(\Phi[v_j])(y) + \sum_{j=1}^m \sum_{k=1}^{q_{ij}} c_{ijk}(y) \epsilon(D'_{ijk}) \partial(D''_{ijk}) \gamma_{0,x}(\Phi[v_j])(y).
\end{aligned}$$

Here the last equality follows from the fact that $\gamma_{0,x}(\Phi[v_j]) = 0$ for $j = m+1, \dots, n$. Now putting $S_{ij} = \sum_{k=1}^{q_{ij}} c_{ijk}(y) \epsilon(D'_{ijk}) \partial(D''_{ijk})$ we get (7.6). Its uniqueness is due to the surjectivity of $\Gamma_{0,x}^V$ (Lemma 5.3). \square

Lemma 7.5. *Suppose $\psi \in \text{Hom}_W(E_{\text{single}}^M, P_{\mathbf{H}}(V_{\text{single}}^M))$ and take $f_{ij} \in S(\mathfrak{a}_{\mathbb{C}})$ so that*

$$(\theta_{\mathbf{H}} \circ \psi)[e_i] = \sum_{j=1}^{m'} f_{ij} \otimes v_j \quad \text{for } i = 1, \dots, \mu'.$$

Then there exists a unique $T = (T_{ij}) \in \text{Mat}(\mu', m'; \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ such that

$$(7.7) \quad (\varphi \circ \psi)[e_i] = \sum_{j=1}^{m'} \mathcal{T}(f_{ij}) \varphi[v_j] = \sum_{j=1}^{m'} \left(\partial(f_{ij}(\cdot - \rho)) + T_{ij} \right) \varphi[v_j]$$

for any $i = 1, \dots, \mu'$ and

$$\varphi \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), C^\infty(Wx)) \simeq \text{Hom}_W(V_{\text{single}}^M, C^\infty(Wx)) \simeq \text{Hom}_{\mathbb{C}}(V_{\text{single}}^M, C_x^\infty).$$

Note $h \in \mathbf{H}$ acts on $C^\infty(Wx)$ by $\mathcal{T}(\theta_{\mathbf{H}} h)$. We consider $\varphi[v_j] \in C^\infty(Wx)$ in the second expression of (7.7) while $\varphi[v_j] \in C_x^\infty$ in the third expression.

Proof. Immediate from the argument just before Lemma 5.6. \square

Now suppose Ψ , D_{ij} and S are as in Lemma 7.4. According to the divisions of bases

$$\begin{aligned}
\{e_1, \dots, e_\mu\} &= \{e_1, \dots, e_{\mu'}\} \sqcup \{e_{\mu'+1}, \dots, e_\mu\}, \\
\{v_1, \dots, v_m\} &= \{v_1, \dots, v_{m'}\} \sqcup \{v_{m'+1}, \dots, v_m\},
\end{aligned}$$

we divide S into four blocks:

$$S = \begin{pmatrix} S_{\text{single}} & P \\ Q & S_{\text{double}} \end{pmatrix}.$$

Similarly we divide the matrix $(\partial(\gamma(D_{ij})(\cdot - \rho)))_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq m}} \in \text{Mat}(\mu, m; \partial(S(\mathfrak{a}_{\mathbb{C}})))$ into four blocks:

$$(\partial(\gamma(D_{ij})(\cdot - \rho)))_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq m}} = \begin{pmatrix} \partial_{\text{single}} & \partial_P \\ \partial_Q & \partial_{\text{double}} \end{pmatrix}.$$

For $\psi := \tilde{\Gamma}_V^E(\Psi)$ let f_{ij} and T be as in Lemma 7.5. Because of (7.2) we clearly have

$$f_{ij} = \gamma(D_{ij}) \quad \text{for } i = 1, \dots, \mu' \text{ and } j = 1, \dots, m',$$

and hence

$$(7.8) \quad (\partial(f_{ij}(\cdot - \rho)))_{\substack{1 \leq i \leq \mu' \\ 1 \leq j \leq m'}} = \partial_{\text{single}}.$$

Let $F_{\gamma(L_{\mathfrak{g}})}$ (resp., $F'_{\gamma(L_{\mathfrak{g}})}$) be the F of Lemma 5.6 for $U = V_{\text{single}}^M$ (resp., $U = E_{\text{single}}^M$) and $\Delta = \gamma(L_{\mathfrak{g}}) \in S(\mathfrak{a}_{\mathbb{C}})^W$. By a result of §5, there exist a matrix $L_{\text{double}} \in \text{Mat}(m - m', m - m'; \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ such that for any $\Phi \in \text{Hom}_K(V^M, C^\infty(K\bar{x}))$

$$\Gamma_{0,x}^V(r(L_{\mathfrak{g}}) \circ \Phi) = \left(\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + \begin{pmatrix} F_{\gamma(L_{\mathfrak{g}})} & 0 \\ 0 & L_{\text{double}} \end{pmatrix} \right) {}^t(\Gamma_{0,x}^V(\Phi)[v_1], \dots, \Gamma_{0,x}^V(\Phi)[v_m]),$$

and a matrix $L'_{\text{double}} \in \text{Mat}(\mu - \mu', \mu - \mu'; \mathcal{M} \otimes \partial(S(\mathfrak{a}_{\mathbb{C}})))$ such that for any $\Phi' \in \text{Hom}_K(E^M, C^\infty(K\bar{x}))$

$$\Gamma_{0,x}^E(r(L_{\mathfrak{g}}) \circ \Phi') = \left(\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + \begin{pmatrix} F'_{\gamma(L_{\mathfrak{g}})} & 0 \\ 0 & L'_{\text{double}} \end{pmatrix} \right) {}^t(\Gamma_{0,x}^E(\Phi')[e_1], \dots, \Gamma_{0,x}^E(\Phi')[e_\mu]).$$

Since $r(\cdot)$ and $\ell(\cdot)$ commute, we have $r(L_{\mathfrak{g}}) \circ (\Phi \circ \Psi) = (r(L_{\mathfrak{g}}) \circ \Phi) \circ \Psi$ for any $\Phi \in \text{Hom}_K(V^M, C^\infty(K\bar{x}))$. Therefore Lemma 7.4 and the surjectivity of $\Gamma_{0,x}^V$ imply the matrix identity:

$$(7.9) \quad \left(\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + \begin{pmatrix} F'_{\gamma(L_{\mathfrak{g}})} & 0 \\ 0 & L'_{\text{double}} \end{pmatrix} \right) \begin{pmatrix} \partial_{\text{single}} + S_{\text{single}} & \partial_P + P \\ \partial_Q + Q & \partial_{\text{double}} + S_{\text{double}} \end{pmatrix} \\ = \begin{pmatrix} \partial_{\text{single}} + S_{\text{single}} & \partial_P + P \\ \partial_Q + Q & \partial_{\text{double}} + S_{\text{double}} \end{pmatrix} \left(\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + \begin{pmatrix} F_{\gamma(L_{\mathfrak{g}})} & 0 \\ 0 & L_{\text{double}} \end{pmatrix} \right).$$

On the other hand, it follows from Proposition 4.3 (i) that $\mathcal{T}(\gamma(L_{\mathfrak{g}})) \circ (\varphi \circ \psi) = (\mathcal{T}(\gamma(L_{\mathfrak{g}})) \circ \varphi) \circ \psi$ for any $\varphi \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), C^\infty(Wx)) \simeq \text{Hom}_{\mathbb{C}}(V_{\text{single}}^M, C_x^\infty)$. Hence Lemma 5.6, Lemma 7.5 and (7.8) imply the matrix identity:

$$(7.10) \quad (\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + F'_{\gamma(L_{\mathfrak{g}})})(\partial_{\text{single}} + T) = (\partial_{\text{single}} + T)(\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + F_{\gamma(L_{\mathfrak{g}})}).$$

Now by comparing the upper-left blocks in (7.9) we get

$$(7.11) \quad (\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + F'_{\gamma(L_{\mathfrak{g}})})(\partial_{\text{single}} + S_{\text{single}}) = (\partial_{\text{single}} + S_{\text{single}})(\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)) + F_{\gamma(L_{\mathfrak{g}})}).$$

Subtracting (7.10) from (7.11),

$$(7.12) \quad [\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), S_{\text{single}} - T] = (S_{\text{single}} - T)F_{\gamma(L_{\mathfrak{g}})} - F'_{\gamma(L_{\mathfrak{g}})}(S_{\text{single}} - T).$$

Applying Lemma 5.7 to (7.12) we conclude

$$(7.13) \quad S_{\text{single}} = T.$$

If $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(K\bar{x}))$, that is, if $\Gamma_{0,x}^V(\Phi)[v_j] = 0$ for $j = m' + 1, \dots, m$, then for $i = 1, \dots, \mu'$

$$\begin{aligned} \tilde{\Gamma}_{0,x}^E(\Phi \circ \Psi)[e_i] &= \Gamma_{0,x}^E(\Phi \circ \Psi)[e_i] && \text{(by definition)} \\ &= \sum_{j=1}^m \left(\partial(\gamma(D_{ij})(\cdot - \rho)) + S_{ij} \right) \Gamma_{0,x}^V(\Phi)[v_j] && (\because \text{Lemma 7.4}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{m'} \left(\partial(\gamma(D_{ij})(\cdot - \rho)) + S_{ij} \right) \tilde{\Gamma}_{0,x}^V(\Phi)[v_j] \\
&= \sum_{j=1}^{m'} \left(\partial(f_{ij}(\cdot - \rho)) + T_{ij} \right) \tilde{\Gamma}_{0,x}^V(\Phi)[v_j] \quad (\because (7.8) \text{ and } (7.13)) \\
&= (\tilde{\Gamma}_{0,x}(\Phi) \circ \tilde{\Gamma}_V^E(\Psi))[e_i]. \quad (\because \text{Lemma 7.5})
\end{aligned}$$

This proves Theorem 7.3 (i).

Next, suppose $\Phi \in \text{Hom}_K(V, C^\infty(K\bar{x}))$ is arbitrary and $\Psi \in \text{Hom}_K^{1 \rightarrow 1}(E, P_G(V))$. Then $\theta \circ \Psi \in \text{Hom}_K^{1 \rightarrow 1}(E, P_G(V))$ by (7.3). Thus $\gamma(D_{ij}) = 0$ for $i = 1, \dots, \mu'$ and $j = m' + 1, \dots, m$ in view of (7.5), which means $\partial_P = 0$. Hence by comparing the upper-right blocks in (7.9) we get

$$[\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), P] = PL_{\text{double}} - F'_{\gamma(L_{\mathfrak{g}})}P.$$

Applying Lemma 5.7 to this, we obtain $P = 0$. Thus for $i = 1, \dots, \mu'$

$$\begin{aligned}
\tilde{\Gamma}_{0,x}^E(\Phi \circ \Psi)[e_i] &= \sum_{j=1}^m \left(\partial(\gamma(D_{ij})(\cdot - \rho)) + S_{ij} \right) \Gamma_{0,x}^V(\Phi)[v_j] \\
&= \sum_{j=1}^{m'} \left(\partial(\gamma(D_{ij})(\cdot - \rho)) + S_{ij} \right) \Gamma_{0,x}^V(\Phi)[v_j] \\
&\quad + \sum_{j=1}^{m-m'} \left((\partial_P)_{ij} + P_{ij} \right) \Gamma_{0,x}^V(\Phi)[v_{m'+j}] \\
&= \sum_{j=1}^{m'} \left(\partial(\gamma(D_{ij})(\cdot - \rho)) + S_{ij} \right) \tilde{\Gamma}_{0,x}^V(\Phi)[v_j].
\end{aligned}$$

Hence the same calculation as before proves Theorem 7.3 (ii).

Finally, suppose $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(K\bar{x}))$ and $\Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, P_G(V))$. Since $\theta \circ \Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, P_G(V))$ by (7.4), $\gamma(D_{ij}) = 0$ for $i = \mu' + 1, \dots, \mu$ and $j = 1, \dots, m'$. This means $\partial_Q = 0$. By comparing the lower-left blocks in (7.9) we get

$$[\partial(\gamma(L_{\mathfrak{g}})(\cdot - \rho)), Q] = QF_{\gamma(L_{\mathfrak{g}})} - L'_{\text{double}}Q$$

and hence $Q = 0$. Since $\Gamma_{0,x}^V(\Phi)[v_j] = 0$ for $j = m' + 1, \dots, m$, we calculate for $i = \mu' + 1, \dots, \mu$

$$\begin{aligned}
\Gamma_{0,x}^E(\Phi \circ \Psi)[e_i] &= \sum_{j=1}^m \left(\partial(\gamma(D_{ij})(\cdot - \rho)) + S_{ij} \right) \Gamma_{0,x}^V(\Phi)[v_j] \\
&= \sum_{j=1}^{m'} \left((\partial_Q)_{i-\mu'+j} + Q_{i-\mu'+j} \right) \Gamma_{0,x}^V(\Phi)[v_j] \\
&= 0.
\end{aligned}$$

This shows $\Phi \circ \Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, C^\infty(K\bar{x}))$, proving Theorem 7.3 (iii).

8. CORRESPONDENCES OF SUBMODULES

In this section we study various correspondences which send \mathbf{H} -modules to $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules. One example is Ξ_0^{\min} which maps an \mathbf{H} -submodule of $C^\infty(A)$ to a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule of $C^\infty(G/K)_{K\text{-finite}}$. Another example is Ξ^{\min} which maps an \mathbf{H} -submodule of $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})$ to a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule of $P_G(\mathbb{C}_{\text{triv}})$. Both correspondences have a nice property on the multiplicities of (quasi-)single-petaled K -types. This property for Ξ_0^{\min} comes from the generalized Chevalley restriction theorem (Theorem 2.3) and the radial part formula in the last section (Theorem 7.3), and for Ξ^{\min} from the generalized Harish-Chandra isomorphism (Theorem 6.5 (ii)) and the functoriality of the generalized Harish-Chandra homomorphism (Proposition 6.3). Motivated by this obvious formal parallelism we shall develop a unified argument by introducing the following three categories \mathcal{C}_{Ch} , $\mathcal{C}_{\text{w-rad}}$ and \mathcal{C}_{rad} :

Definition 8.1 (the category \mathcal{C}_{Ch}). An object of \mathcal{C}_{Ch} is a pair $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ of a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module \mathcal{M}_G and an \mathbf{H} -module $\mathcal{M}_{\mathbf{H}}$ satisfying: all the K -types of \mathcal{M}_G belong to \widehat{K}_M ; to each $V \in \widehat{K}_M$ there attach a linear map $\tilde{\Gamma}_{\mathcal{M}}^V : \text{Hom}_K(V, \mathcal{M}_G) \rightarrow \text{Hom}_W(V_{\text{single}}^M, \mathcal{M}_{\mathbf{H}})$ and a linear subspace $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$ of $\text{Hom}_K(V, \mathcal{M}_G)$ such that the restriction of $\tilde{\Gamma}_{\mathcal{M}}^V$ to $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$ gives a bijection

$$(\text{Ch-0}) \quad \tilde{\Gamma}_{\mathcal{M}}^V : \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \xrightarrow{\sim} \text{Hom}_W(V_{\text{single}}^M, \mathcal{M}_{\mathbf{H}}).$$

Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$, $\mathcal{N} = (\mathcal{N}_G, \mathcal{N}_{\mathbf{H}}) \in \mathcal{C}_{\text{Ch}}$. Then a morphism of \mathcal{C}_{Ch} between them is a pair $\mathcal{I} = (\mathcal{I}_G, \mathcal{I}_{\mathbf{H}})$ of a $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I}_G : \mathcal{M}_G \rightarrow \mathcal{N}_G$ and an \mathbf{H} -homomorphism $\mathcal{I}_{\mathbf{H}} : \mathcal{M}_{\mathbf{H}} \rightarrow \mathcal{N}_{\mathbf{H}}$ which satisfies the following two conditions:

(Ch-1) For any $V \in \widehat{K}_M$ the diagram

$$\begin{array}{ccc} \text{Hom}_K(V, \mathcal{M}_G) & \xrightarrow{\mathcal{I}_G \circ \cdot} & \text{Hom}_K(V, \mathcal{N}_G) \\ \tilde{\Gamma}_{\mathcal{M}}^V \downarrow & & \downarrow \tilde{\Gamma}_{\mathcal{N}}^V \\ \text{Hom}_W(V_{\text{single}}^M, \mathcal{M}_{\mathbf{H}}) & \xrightarrow{\mathcal{I}_{\mathbf{H}} \circ \cdot} & \text{Hom}_W(V_{\text{single}}^M, \mathcal{N}_{\mathbf{H}}) \end{array}$$

commutes.

(Ch-2) For any $V \in \widehat{K}_M$ and $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$, $\mathcal{I}_G \circ \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{N}_G)$, namely, the following map is well defined:

$$\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \xrightarrow{\mathcal{I}_G \circ \cdot} \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{N}_G).$$

Remark 8.2. Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{Ch}}$. For each $V \in \widehat{K}_M$ we have the direct sum decomposition

$$\text{Hom}_K(V, \mathcal{M}_G) = \text{Ker } \tilde{\Gamma}_{\mathcal{M}}^V \oplus \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G).$$

For $V \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$ it necessarily holds that $\tilde{\Gamma}_{\mathcal{M}}^V = 0$ and $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) = \{0\}$. Hence any pair of a $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I}_G : \mathcal{M}_G \rightarrow \mathcal{N}_G$ and an \mathbf{H} -homomorphism $\mathcal{I}_{\mathbf{H}} : \mathcal{M}_{\mathbf{H}} \rightarrow \mathcal{N}_{\mathbf{H}}$ automatically satisfies Conditions (Ch-1) and (Ch-2) for $V \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$.

Definition 8.3 (the category $\mathcal{C}_{\text{w-rad}}$). We call an object of $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{Ch}}$ a *weak radial pair* if it satisfies

(w-rad) for any $E, V \in \widehat{K}_M$, $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(V), \mathcal{M}_G)$, and $\Psi \in \text{Hom}_W(E, P_G(V))$ it holds that

$$\tilde{\Gamma}_{\mathcal{M}}^E(\Phi \circ \Psi) = \tilde{\Gamma}_{\mathcal{M}}(\Phi) \circ \tilde{\Gamma}_V^E(\Psi).$$

Here $\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(V), \mathcal{M}_G)$ denotes the subspace of $\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \mathcal{M}_G)$ corresponding to $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$ under the identification $\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \mathcal{M}_G) \simeq \text{Hom}_K(V, \mathcal{M}_G)$ and

$$\tilde{\Gamma}_{\mathcal{M}} : \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \mathcal{M}_G) \rightarrow \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{M}_{\mathbf{H}})$$

is the map identified with

$$\tilde{\Gamma}_{\mathcal{M}}^V : \text{Hom}_K(V, \mathcal{M}_G) \longrightarrow \text{Hom}_W(V_{\text{single}}^M, \mathcal{M}_{\mathbf{H}}).$$

The category $\mathcal{C}_{\text{w-rad}}$ is the full subcategory of \mathcal{C}_{Ch} consisting of the weak radial pairs.

Remark 8.4. If $E \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$ or $V \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$ then the formula in (w-rad) is automatic.

Definition 8.5 (the category \mathcal{C}_{rad}). We call an object of $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{w-rad}}$ a *radial pair* if it satisfies the following two additional conditions:

(rad-1) For any $E, V \in \widehat{K}_M$, $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \mathcal{M}_G)$, and $\Psi \in \text{Hom}_W^{1 \rightarrow 1}(E, P_G(V))$ it holds that

$$\tilde{\Gamma}_{\mathcal{M}}^E(\Phi \circ \Psi) = \tilde{\Gamma}_{\mathcal{M}}(\Phi) \circ \tilde{\Gamma}_V^E(\Psi).$$

(rad-2) For any $E, V \in \widehat{K}_M$, $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(V), \mathcal{M}_G)$, and $\Psi \in \text{Hom}_W^{2 \rightarrow 2}(E, P_G(V))$ it holds that

$$\Phi \circ \Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, \mathcal{M}_G).$$

The category \mathcal{C}_{rad} is the full subcategory of $\mathcal{C}_{\text{w-rad}}$ consisting of the radial pairs.

Remark 8.6. If $E \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$ then the formula in (rad-1) is automatic. If $V \in \widehat{K}_M \setminus \widehat{K}_{\text{qsp}}$ then the condition in (rad-2) is automatic.

Definition 8.7 (radial restrictions). Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{Ch}}$. We say a linear map $\gamma_{\mathcal{M}} : \mathcal{M}_G \rightarrow \mathcal{M}_{\mathbf{H}}$ is a *radial restriction* of \mathcal{M} if it satisfies the following two conditions:

(rest-1) For any $V \in \widehat{K}_M$

$$\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) = \{\Phi \in \text{Hom}_K(V, \mathcal{M}_G); (\gamma_{\mathcal{M}} \circ \Phi)[V_{\text{double}}^M] = \{0\}\}.$$

(rest-2) For any $V \in \widehat{K}_M$, $\tilde{\Gamma}_{\mathcal{M}}^V$ coincides with the linear map

$$\text{Hom}_K(V, \mathcal{M}_G) \ni \Phi \longmapsto (V_{\text{single}}^M \hookrightarrow V \xrightarrow{\Phi} \mathcal{M}_G \xrightarrow{\gamma_{\mathcal{M}}} \mathcal{M}_{\mathbf{H}}) \in \text{Hom}_{\mathbb{C}}(V_{\text{single}}^M, \mathcal{M}_{\mathbf{H}}).$$

(Hence actually the rightmost part can be replaced by Hom_W .)

Note $\gamma_{\mathcal{M}}$ completely determines the structure of \mathcal{M} .

Remark 8.8. Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{Ch}}$ has a radial restriction $\gamma_{\mathcal{M}}$. Then $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) = \text{Hom}_K(V, \mathcal{M}_G)$ for $V \in \widehat{K}_{\text{sp}}$. For each $V \in \widehat{K}_M$ we can define in addition to $\tilde{\Gamma}_{\mathcal{M}}^V$ a linear map

$$(8.1) \quad \begin{aligned} \Gamma_{\mathcal{M}}^V : \text{Hom}_K(V, \mathcal{M}_G) &\longrightarrow \text{Hom}_{\mathbb{C}}(V^M, \mathcal{M}_{\mathbf{H}}); \\ \Phi &\longmapsto (\varphi : V^M \hookrightarrow V \xrightarrow{\Phi} \mathcal{M}_G \xrightarrow{\gamma_{\mathcal{M}}} \mathcal{M}_{\mathbf{H}}). \end{aligned}$$

This is injective since for $\Phi \in \text{Hom}_K(V, \mathcal{M}_G)$ we have

$$\Gamma_{\mathcal{M}}^V(\Phi) = 0 \iff \tilde{\Gamma}_{\mathcal{M}}^V(\Phi) = 0 \text{ and } \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \iff \Phi = 0.$$

Let us look at some examples:

Proposition 8.9. $((\ell(\cdot), C^\infty(G/K)_{K\text{-finite}}), (\mathcal{T}(\theta_{\mathbf{H}}\cdot), C^\infty(A)))$ is a radial pair with radial restriction γ_0 .

Proof. It is well known (or easy to show) that all the K -types of $C^\infty(G/K)_{K\text{-finite}}$ belong to \widehat{K}_M . The other conditions are satisfied by Theorem 2.3, (2.3) and Theorem 7.3. \square

Recall the Harish-Chandra homomorphism $\gamma : P_G(\mathbb{C}_{\text{triv}}) \rightarrow P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})$ in Example 6.1.

Proposition 8.10. $(P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$ is a radial pair with radial restriction γ .

Proof. Recall \mathfrak{s} is the vector part of the Cartan decomposition of \mathfrak{g} . If \mathbf{S} denotes the image of the symmetrization map of $S(\mathfrak{s}_{\mathbb{C}})$ then one has $P_G(\mathbb{C}_{\text{triv}}) \simeq \mathbf{S} \otimes v_{\text{triv}}$. Thus a K -type of $P_G(\mathbb{C}_{\text{triv}})$ is a K -type of $S(\mathfrak{s}_{\mathbb{C}})$ and hence is belonging to \widehat{K}_M [KR]. The other conditions follow from Theorem 6.5 (i), (ii) and Proposition 6.3. \square

Proposition 8.11. Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{w-rad}}$ satisfies (rad-2) and $\mathcal{M}_{\mathbf{H}}$ has a W -invariant element ϕ_W . Fix a non-zero $v_{\text{triv}} \in \mathbb{C}_{\text{triv}}$. Then there exists a unique K -invariant element ϕ_K in \mathcal{M}_G such that under the bijection

$$\tilde{\Gamma}_{\mathcal{M}}^{\mathbb{C}_{\text{triv}}} : \text{Hom}_K^{2 \rightarrow 2}(\mathbb{C}_{\text{triv}}, \mathcal{M}_G) \xrightarrow{\sim} \text{Hom}_W(\mathbb{C}_{\text{triv}}, \mathcal{M}_{\mathbf{H}})$$

$(\mathbb{C}_{\text{triv}} \ni c v_{\text{triv}} \mapsto c \phi_K \in \mathcal{M}_G)$ corresponds to $(\mathbb{C}_{\text{triv}} \ni c v_{\text{triv}} \mapsto c \phi_W \in \mathcal{M}_{\mathbf{H}})$. (If \mathcal{M} has a radial restriction $\gamma_{\mathcal{M}}$ then ϕ_K is a unique K -invariant element such that $\gamma_{\mathcal{M}}(\phi_K) = \phi_W$.) If we define

$$\begin{aligned} \mathcal{I}_G : P_G(\mathbb{C}_{\text{triv}}) &= U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathbb{C}_{\text{triv}} \ni D \otimes v_{\text{triv}} \longmapsto D \phi_K \in \mathcal{M}_G, \\ \mathcal{I}_{\mathbf{H}} : P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}) &= \mathbf{H} \otimes_{\mathbb{C}W} \mathbb{C}_{\text{triv}} \ni h \otimes v_{\text{triv}} \longmapsto h \phi_W \in \mathcal{M}_{\mathbf{H}} \end{aligned}$$

then $\mathcal{I} = (\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : (P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})) \rightarrow \mathcal{M}$ is a morphism of $\mathcal{C}_{\text{w-rad}}$.

Proof. Let $\Phi := (\mathbb{C}_{\text{triv}} \ni c v_{\text{triv}} \mapsto c \phi_K \in \mathcal{M}_G) \in \text{Hom}_K^{2 \rightarrow 2}(\mathbb{C}_{\text{triv}}, \mathcal{M}_G)$. Thus $\tilde{\Gamma}_{\mathcal{M}}^{\mathbb{C}_{\text{triv}}}(\Phi) = (\mathbb{C}_{\text{triv}} \ni c v_{\text{triv}} \mapsto c \phi_W \in \mathcal{M}_{\mathbf{H}}) \in \text{Hom}_W(\mathbb{C}_{\text{triv}}, \mathcal{M}_{\mathbf{H}})$. Now for $V \in \widehat{K}_M$ and $\Psi \in \text{Hom}_K(V, P_G(\mathbb{C}_{\text{triv}}))$ we have

$$\begin{aligned} \mathcal{I}_G \circ \Psi &= \Phi \circ \Psi \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \quad \text{if } \Psi \in \text{Hom}_K^{2 \rightarrow 2}, & (\because \text{rad-2}) \\ \tilde{\Gamma}_{\mathcal{M}}^V(\mathcal{I}_G \circ \Psi) &= \tilde{\Gamma}_{\mathcal{M}}^V(\Phi \circ \Psi) = \tilde{\Gamma}_{\mathcal{M}}(\Phi) \circ \tilde{\Gamma}_{\mathbb{C}_{\text{triv}}}^V(\Psi), & (\because \text{w-rad}) \end{aligned}$$

$$\mathcal{I}_{\mathbf{H}} \circ \tilde{\Gamma}_{\mathbb{C}_{\text{triv}}}^V(\Psi) = \tilde{\Gamma}_{\mathcal{M}}(\Phi) \circ \tilde{\Gamma}_{\mathbb{C}_{\text{triv}}}^V(\Psi).$$

The first property shows \mathcal{I} satisfies (Ch-2). The second and third ones show \mathcal{I} satisfies (Ch-1). \square

One can check the following basic properties of our categories in a straightforward way.

Proposition 8.12. *The categories \mathcal{C}_{Ch} , $\mathcal{C}_{\text{w-rad}}$ and \mathcal{C}_{rad} are Abelian categories. Suppose $\mathcal{I} = (\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : \mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \rightarrow \mathcal{N} = (\mathcal{N}_G, \mathcal{N}_{\mathbf{H}})$ is a morphism in \mathcal{C}_{Ch} .*

(i) *\mathcal{I} is mono if and only if both \mathcal{I}_G and $\mathcal{I}_{\mathbf{H}}$ are injective. \mathcal{I} is epi if and only if both \mathcal{I}_G and $\mathcal{I}_{\mathbf{H}}$ are surjective.*

(ii) *Put $\mathcal{K}_G = \text{Ker } \mathcal{I}_G$ and $\mathcal{K}_{\mathbf{H}} = \text{Ker } \mathcal{I}_{\mathbf{H}}$. Then we can identify $\text{Ker } \mathcal{I} \simeq \mathcal{K} := (\mathcal{K}_G, \mathcal{K}_{\mathbf{H}})$ where for each $V \in \hat{K}_M$*

$$\begin{aligned} \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{K}_G) &= \text{Hom}_K(V, \mathcal{K}_G) \cap \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \\ &= \{ \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G); \tilde{\Gamma}_{\mathcal{M}}^V(\Phi) \in \text{Hom}_W(V_{\text{single}}^M, \mathcal{K}_{\mathbf{H}}) \}. \end{aligned}$$

If \mathcal{M} satisfies (w-rad) (that is $\mathcal{M} \in \mathcal{C}_{\text{w-rad}}$) then so does \mathcal{K} . The same thing holds for (rad-1) or (rad-2).

(iii) *Put $\mathcal{Q}_G = \text{Coker } \mathcal{I}_G$ and $\mathcal{Q}_{\mathbf{H}} = \text{Coker } \mathcal{I}_{\mathbf{H}}$. Then we can identify $\text{Coker } \mathcal{I} \simeq \mathcal{Q} := (\mathcal{Q}_G, \mathcal{Q}_{\mathbf{H}})$ where for each $V \in \hat{K}_M$*

$$\begin{aligned} \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{Q}_G) &= \text{the image of } \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{N}_G) \\ &\text{under } \text{Hom}_K(V, \mathcal{N}_G) \rightarrow \text{Hom}_K(V, \mathcal{Q}_G). \end{aligned}$$

If \mathcal{N} satisfies (w-rad) (that is $\mathcal{N} \in \mathcal{C}_{\text{w-rad}}$) then so does \mathcal{Q} . The same thing holds for (rad-1) or (rad-2).

The next lemma will be repeatedly used:

Lemma 8.13. *Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{Ch}}$ and a linear map $\gamma_{\mathcal{M}} : \mathcal{M}_G \rightarrow \mathcal{M}_{\mathbf{H}}$ satisfies Condition (rest-2) and the following two conditions:*

(rest-1') *For any $V \in \hat{K}_M$*

$$\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \subset \{ \Phi \in \text{Hom}_K(V, \mathcal{M}_G); (\gamma_{\mathcal{M}} \circ \Phi)[V_{\text{double}}^M] = \{0\} \}.$$

(rest-3) *If $y \in \mathcal{M}_G$ then*

$$(8.2) \quad \gamma_{\mathcal{M}}(Hy) = (H + \rho(H))\gamma_{\mathcal{M}}(y) \quad \text{for } H \in \mathfrak{a}_{\mathbb{C}},$$

$$(8.3) \quad \gamma_{\mathcal{M}}(Xy) = 0 \quad \text{for } X \in \mathfrak{n}_{\mathbb{C}},$$

$$(8.4) \quad \gamma_{\mathcal{M}}(my) = 0 \quad \text{for } m \in M.$$

Then \mathcal{M} is a weak radial pair satisfying (rad-1). Moreover, if $\gamma_{\mathcal{M}}$ satisfies (rest-1) then \mathcal{M} is a radial pair.

Remark 8.14. In the setting of the lemma let $\gamma^{\mathcal{M}_G} : \mathcal{M}_G \rightarrow \Gamma(\mathcal{M}_G)$ be the linear map defined by (6.1). Then Condition (rest-3) holds if and only if there exists an $\mathfrak{a}_{\mathbb{C}}$ -homomorphism $\delta^{\mathcal{M}_G} : \Gamma(\mathcal{M}_G) \rightarrow \mathcal{M}_{\mathbf{H}}$ such that $\gamma_{\mathcal{M}} = \delta^{\mathcal{M}_G} \circ \gamma^{\mathcal{M}_G}$.

Example 8.15. The radial restriction γ for the radial pair $(P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$ satisfies (rest-3) since $\gamma^{P_G(\mathbb{C}_{\text{triv}})} = \gamma$.

Proof of Lemma 8.13. Suppose $E, V \in \widehat{K}_M$ and let $\{v_1, \dots, v_{m'}\}$, $\{v_{m'+1}, \dots, v_m\}$ and $\{v_{m+1}, \dots, v_n\}$ be bases of V_{single}^M , V_{double}^M and $(V^M)^\perp$. Let $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \mathcal{M}_G)$, $\Psi \in \text{Hom}_K(E, P_G(V))$. For any $e \in E^M$ let

$$\Psi[e] = \sum_{i=1}^n D_i \otimes v_i \quad \text{with } D \in U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}}).$$

Then $\Gamma_V^E(\Psi)[e] = \sum_{i=1}^m \gamma(D_i) \otimes v_i$ and
(8.5)

$$\begin{aligned} (\gamma_{\mathcal{M}} \circ \Phi \circ \Psi)[e] &= \sum_{i=1}^n \gamma_{\mathcal{M}}(D_i \Phi[v_i]) \\ &= \sum_{i=1}^n \gamma(D_i) \gamma^{\mathcal{M}}(\Phi[v_i]) & (\because (8.2), (8.3)) \\ &= \sum_{i=1}^m \gamma(D_i) \gamma^{\mathcal{M}}(\Phi[v_i]) & (\because (8.4)) \\ &= \sum_{i=1}^{m'} \gamma(D_i) \tilde{\Gamma}_{\mathcal{M}}(\Phi)[v_i] + \sum_{i=m'+1}^m \gamma(D_i) (\gamma^{\mathcal{M}} \circ \Phi)[v_i]. & (\because (\text{rest-2})) \end{aligned}$$

Here the first summand is zero when $\Psi \in \text{Hom}_K^{2 \rightarrow 2}$ and $e \in E_{\text{double}}^M$ since in this case $\gamma(D_i) = 0$ for $i = 1, \dots, m'$. Also, the second summand is zero when $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}$ since in this case $(\gamma^{\mathcal{M}} \circ \Phi)[v_i] = 0$ for $i = m' + 1, \dots, m$ by (rest-1').

Hence if $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}$ and $\Psi \in \text{Hom}_K^{2 \rightarrow 2}$ then

$$\gamma_{\mathcal{M}} \circ (\Phi \circ \Psi)[V_{\text{double}}^M] = \{0\}.$$

This means (rest-1) implies (rad-2) under the assumption of the lemma.

Now suppose $e \in E_{\text{single}}^M$. Then (8.5) reduces to

$$\tilde{\Gamma}_{\mathcal{M}}^E(\Phi \circ \Psi)[e] = (\tilde{\Gamma}_{\mathcal{M}}(\Phi) \circ \tilde{\Gamma}_V^E(\Psi))[e] + \sum_{i=m'+1}^m \gamma(D_i) (\gamma^{\mathcal{M}} \circ \Phi)[v_i].$$

Since the second summand is zero when $\Phi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}$, Condition (w-rad) is satisfied. The second summand also vanishes when $\Psi \in \text{Hom}_K^{1 \rightarrow 1}$ since in this case $\gamma(D_i) = 0$ for $i = m' + 1, \dots, m$. Thus Condition (rad-1) is satisfied. \square

In this paper we shall introduce various functors connecting the category $\mathbf{H}\text{-Mod}$ of \mathbf{H} -modules to the category $(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ of $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules. Each one is a descendant of the following:

Definition 8.16 (the functor $\Xi_{\text{w-rad}}$). For each $V \in \widehat{K}_M$ we define a $(\mathfrak{g}_{\mathbb{C}}, K)$ -subspace $Q_G(V)$ of $P_G(V)$ by

$$Q_G(V) = \sum_{E \in \widehat{K} \setminus \widehat{K}_M} U(\mathfrak{g}_{\mathbb{C}}) \text{ (the } E\text{-isotypic component of } P_G(V)\text{)}$$

and put $\bar{P}_G(V) := P_G(V)/Q_G(V)$. For $\mathcal{X} \in \mathbf{H}\text{-Mod}$ we define a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module

$$\begin{aligned} \Xi_{\text{w-rad}}(\mathcal{X}) &= \bigoplus_{V \in \widehat{K}_M} \bar{P}_G(V) \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) \\ &\simeq \bigoplus_{V \in \widehat{K}_M} \bar{P}_G(V) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X}) \end{aligned}$$

where $\mathfrak{g}_{\mathbb{C}}$ and K act only on the $\bar{P}_G(V)$ -parts. The correspondence $\Xi_{\text{w-rad}} : \mathbf{H}\text{-Mod} \rightarrow (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ clearly defines an exact functor .

The next lemma accumulates easy properties of $\bar{P}_G(V)$.

Lemma 8.17. Suppose $F, V \in \widehat{K}_M$.

- (i) All the K -types of $\bar{P}_G(V)$ belong to \widehat{K}_M . The right exact functor Γ defined in §6 maps $Q_G(V)$ to 0. Hence $\Gamma(\bar{P}_G(V)) = \Gamma(P_G(V)) = P_{\mathbf{H}}(V^M)$ and the map $\gamma^{P_G(V)} : P_G(V) \rightarrow P_{\mathbf{H}}(V^M)$ defined by (6.1) factors through $\gamma^{\bar{P}_G(V)} : \bar{P}_G(V) \rightarrow P_{\mathbf{H}}(V^M)$.
- (ii) If all the K -types of a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module \mathcal{Y} belong to \widehat{K}_M then

$$\text{Hom}_K(V, \mathcal{Y}) \simeq \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \mathcal{Y}) \simeq \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(\bar{P}_G(V), \mathcal{Y}).$$

- (iii) The surjective map

$$(8.6) \quad \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), P_G(F)) \twoheadrightarrow \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(\bar{P}_G(V), \bar{P}_G(F))$$

is naturally induced from (and is identified with) the surjective map

$$(8.7) \quad \text{Hom}_K(V, P_G(F)) \twoheadrightarrow \text{Hom}_K(V, \bar{P}_G(F)).$$

- (iv) Let $\tilde{\gamma}^{\bar{P}_G(F)} : \bar{P}_G(F) \rightarrow P_{\mathbf{H}}(F_{\text{single}}^M)$ be the composition of $\gamma^{\bar{P}_G(F)} : \bar{P}_G(F) \rightarrow P_{\mathbf{H}}(F^M)$ and the projection $P_{\mathbf{H}}(F^M) = P_{\mathbf{H}}(F_{\text{single}}^M) \oplus P_{\mathbf{H}}(F_{\text{double}}^M) \rightarrow P_{\mathbf{H}}(F_{\text{single}}^M)$. Then the linear map $\tilde{\Gamma}_F^V : \text{Hom}_K(V, P_G(F)) \rightarrow \text{Hom}_W(V_{\text{single}}^M, P_{\mathbf{H}}(F_{\text{single}}^M))$ equals the composition of (8.7) and the linear map $\tilde{\Gamma}_F^V : \text{Hom}_K(V, \bar{P}_G(F)) \rightarrow \text{Hom}_{\mathbb{C}}(V_{\text{single}}^M, P_{\mathbf{H}}(F_{\text{single}}^M))$ defined by

$$\Psi \longmapsto (V_{\text{single}}^M \hookrightarrow V \xrightarrow{\Psi} \bar{P}_G(F) \xrightarrow{\tilde{\gamma}^{\bar{P}_G(F)}} P_{\mathbf{H}}(F_{\text{single}}^M)).$$

Hence $\tilde{\Gamma}_F^V(\Psi) \in \text{Hom}_W$ for any Ψ .

We give the pair $(\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X})$ a structure as an object of $\mathcal{C}_{\text{w-rad}}$. For each $V \in \widehat{K}_M$ we identify

$$\text{Hom}_K(V, \Xi_{\text{w-rad}}(\mathcal{X})) \simeq \bigoplus_{F \in \widehat{K}_M} \text{Hom}_K(V, \bar{P}_G(F)) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(F_{\text{single}}^M), \mathcal{X})$$

and define the linear map $\tilde{\Gamma}_{\text{w-rad}}^V : \text{Hom}_K(V, \Xi_{\text{w-rad}}(\mathcal{X})) \rightarrow \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})$ by

$$(8.8) \quad \tilde{\Gamma}_{\text{w-rad}}^V : \sum_{F \in \hat{K}_M} \sum_{i=1}^{q_F} \Psi_F^i \otimes \varphi_F^i \mapsto \sum_{F \in \hat{K}_M} \sum_{i=1}^{q_F} \varphi_F^i \circ \tilde{\Gamma}_F^V(\Psi_F^i).$$

Moreover for each $V \in \hat{K}_M$ we put

$$\text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{w-rad}}(\mathcal{X})) = I_V \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X})$$

where $I_V \in \text{Hom}_K(V, \bar{P}_G(V))$ denotes the map $V \ni v \mapsto 1 \otimes v \bmod Q_G(V) \in \bar{P}_G(V)$. Then clearly $(\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{Ch}}$ by these data.

We define a linear map

$$\gamma_{\text{w-rad}} : \Xi_{\text{w-rad}}(\mathcal{X}) = \bigoplus_{F \in \hat{K}_M} \bar{P}_G(F) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(F_{\text{single}}^M), \mathcal{X}) \longrightarrow \mathcal{X}$$

by

$$\bar{P}_G(F) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(F_{\text{single}}^M), \mathcal{X}) \ni D \otimes \varphi \mapsto \varphi(\tilde{\gamma}^{\bar{P}_G(F)}(D)) \in \mathcal{X}.$$

One can easily observe

Lemma 8.18. *For $(\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{Ch}}$ the linear map $\gamma_{\text{w-rad}}$ satisfies (rest-1'), (rest-2) and (rest-3).*

Hence from Lemma 8.13 we have

Proposition 8.19. *For $\mathcal{X} \in \mathbf{H}\text{-Mod}$, $(\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X})$ is a weak radial pair satisfying (rad-1).*

We thus get the exact functor $\mathbf{H}\text{-Mod} \ni \mathcal{X} \mapsto (\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{w-rad}}$, which has the following universal property:

Proposition 8.20. *The functor $\mathcal{X} \mapsto (\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X})$ is left adjoint to the functor $\mathcal{C}_{\text{w-rad}} \ni (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \mapsto \mathcal{M}_{\mathbf{H}} \in \mathbf{H}\text{-Mod}$. More precisely, if $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{w-rad}}$ and an \mathbf{H} -homomorphism $\mathcal{I}_{\mathbf{H}} : \mathcal{X} \rightarrow \mathcal{M}_{\mathbf{H}}$ are given, then there exists a unique $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I}_G : \Xi_{\text{w-rad}}(\mathcal{X}) \rightarrow \mathcal{M}_G$ such that $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : (\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X}) \rightarrow (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ is a morphism of $\mathcal{C}_{\text{w-rad}}$.*

Proof. Using (Ch-0) for \mathcal{M} and Lemma 8.17 (ii), we define $\mathcal{I}_G : \Xi_{\text{w-rad}}(\mathcal{X}) \rightarrow \mathcal{M}_G$ by

$$\begin{aligned} \bigoplus_{F \in \hat{K}_M} \bar{P}_G(F) \otimes \text{Hom}_W(F_{\text{single}}^M, \mathcal{X}) &\rightarrow \bigoplus_{F \in \hat{K}_M} \bar{P}_G(F) \otimes \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(\bar{P}_G(F), \mathcal{M}_G) \rightarrow \mathcal{M}_G; \\ D_F \otimes \varphi_F &\mapsto D_F \otimes \Phi_F \text{ with } \tilde{\Gamma}_{\mathcal{M}}(\Phi_F) = \mathcal{I}_{\mathbf{H}} \circ \varphi_F \mapsto \Phi_F(D_F). \end{aligned}$$

This is clearly a $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism. Now suppose $V \in \hat{K}_M$ and express any $\Phi \in \text{Hom}_K(V, \Xi_{\text{w-rad}}(\mathcal{X}))$ as

$$(8.9) \quad \Phi = \sum_{F \in \hat{K}_M} \sum_{i=1}^{q_F} \bar{\Psi}_F^i \otimes \varphi_F^i \quad \text{with} \quad \begin{cases} \Psi_F^i \in \text{Hom}_K(V, P_G(F)), \\ \varphi_F^i \in \text{Hom}_W(F_{\text{single}}^M, \mathcal{X}), \end{cases}$$

where $\bar{\Psi}_F^i$ is the image of Ψ_F^i under (8.7). For each φ_F^i take $\Phi_F^i \in \text{Hom}_K^{2 \rightarrow 2}(F, \mathcal{M}_G)$ so that $\tilde{\Gamma}_{\mathcal{M}}^F(\Phi_F^i) = \mathcal{I}_{\mathbf{H}} \circ \varphi_F^i$. Then

$$\begin{aligned}
\tilde{\Gamma}_{\mathcal{M}}^V(\mathcal{I}_G \circ \Phi) &= \tilde{\Gamma}_{\mathcal{M}}^V \left(\sum_{F \in \hat{K}_M} \sum_{i=1}^{q_F} \Phi_F^i \circ \Psi_F^i \right) \\
&= \sum_{F \in \hat{K}_M} \sum_{i=1}^{q_F} \tilde{\Gamma}_{\mathcal{M}}(\Phi_F^i) \circ \tilde{\Gamma}_F^V(\Psi_F^i) \quad (\because \text{(w-rad) for } \mathcal{M}) \\
&= \mathcal{I}_{\mathbf{H}} \circ \left(\sum_{F \in \hat{K}_M} \sum_{i=1}^{q_F} \varphi_F^i \circ \tilde{\Gamma}_F^V(\Psi_F^i) \right) \\
&= \mathcal{I}_{\mathbf{H}} \circ \left(\sum_{F \in \hat{K}_M} \sum_{i=1}^{q_F} \varphi_F^i \circ \tilde{\Gamma}_F^V(\bar{\Psi}_F^i) \right) \quad (\because \text{Lemma 8.17 (iv)}) \\
&= \mathcal{I}_{\mathbf{H}} \circ \tilde{\Gamma}_{\text{w-rad}}^V(\Phi), \quad (\because (8.8))
\end{aligned}$$

proving (Ch-1) for $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}})$. If $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{w-rad}}(\mathcal{X}))$ then $\Phi = I_V \otimes \varphi_V$ with some $\varphi_V \in \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})$. In this case, by taking a unique element $\Phi_V \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$ such that $\tilde{\Gamma}_{\mathcal{M}}^V(\Phi_V) = \mathcal{I}_{\mathbf{H}} \circ \varphi_V$, we have $\mathcal{I}_G \circ \Phi = \Phi_V$. This shows $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}})$ satisfies (Ch-2). Finally, to prove the uniqueness of \mathcal{I}_G , assume $(\mathcal{I}'_G, \mathcal{I}_{\mathbf{H}}) : (\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X}) \rightarrow (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ is a morphism of $\mathcal{C}_{\text{w-rad}}$. For each $V \in \hat{K}_M$, $V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})$ is contained in the V -isotypic component of $\Xi_{\text{w-rad}}(\mathcal{X})$ and

$$\text{Hom}_K(V, V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})) = \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{w-rad}}(\mathcal{X}))$$

by definition. It follows that \mathcal{I}'_G on $V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})$ is determined by $\mathcal{I}_{\mathbf{H}}$, and hence is equal to \mathcal{I}_G . But since $\Xi_{\text{w-rad}}(\mathcal{X})$ is spanned by $\bigcup_{V \in \hat{K}_M} V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})$, we conclude $\mathcal{I}'_G = \mathcal{I}_G$. \square

Now suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{w-rad}}$ and let us define two correspondences $\Xi_{\mathcal{M}}^{\min}$ and $\Xi_{\mathcal{M}}^{\max}$ sending \mathbf{H} -submodules of $\mathcal{M}_{\mathbf{H}}$ to $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodules of \mathcal{M}_G .

Definition 8.21 (the correspondence $\Xi_{\mathcal{M}}^{\min}$). Suppose \mathcal{X} is an \mathbf{H} -submodule of $\mathcal{M}_{\mathbf{H}}$. For any $V \in \hat{K}_M$ the V -isotypic component of \mathcal{M}_G is naturally identified with $V \otimes \text{Hom}_K(V, \mathcal{M}_G)$. Using (Ch-0) we can think of $\text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) \subset \text{Hom}_W(V_{\text{single}}^M, \mathcal{M}_{\mathbf{H}})$ as a subspace of $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$. Thus

$$V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) \subset V \otimes \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \subset V \otimes \text{Hom}_K(V, \mathcal{M}_G) \subset \mathcal{M}_G.$$

Now we set

$$\begin{aligned}
\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) &= \text{the } U(\mathfrak{g}_{\mathbb{C}})\text{-span of } \sum_{V \in \hat{K}_M} V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) \\
&= \sum_{V \in \hat{K}_M} U(\mathfrak{g}_{\mathbb{C}})(V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})).
\end{aligned}$$

This is a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule of \mathcal{M}_G .

If $\mathcal{M} = (C^\infty(G/K), C^\infty(A))$, we write Ξ_0^{\min} for $\Xi_{\mathcal{M}}^{\min}$. If $\mathcal{M} = (P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$, we write Ξ^{\min} for $\Xi_{\mathcal{M}}^{\min}$.

Definition 8.22 (the correspondence $\Xi_{\mathcal{M}}^{\max}$). Suppose \mathcal{X} is an \mathbf{H} -submodule of $\mathcal{M}_{\mathbf{H}}$. We define a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule $\Xi_{\mathcal{M}}^{\max}(\mathcal{X}) \subset \mathcal{M}_G$ by

$$\Xi_{\mathcal{M}}^{\max}(\mathcal{X}) = \sum \left\{ \begin{array}{l} \text{a } (\mathfrak{g}_{\mathbb{C}}, K)\text{-submodule such that} \\ \mathcal{Y} \subset \mathcal{M}_G; \quad \tilde{\Gamma}_{\mathcal{M}}^V(\text{Hom}_K(V, \mathcal{Y})) \subset \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) \\ \text{for any } V \in \hat{K}_M \end{array} \right\}.$$

If $\mathcal{M} = (C^\infty(G/K), C^\infty(A))$, we write Ξ_0^{\max} for $\Xi_{\mathcal{M}}^{\max}$. If $\mathcal{M} = (P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$, we write Ξ^{\max} for $\Xi_{\mathcal{M}}^{\max}$.

Theorem 8.23. *Retain the setting of the above definitions.*

- (i) $\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) \subset \Xi_{\mathcal{M}}^{\max}(\mathcal{X})$.
- (ii) Suppose a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule $\mathcal{Y} \subset \mathcal{M}_G$ is such that $\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) \subset \mathcal{Y} \subset \Xi_{\mathcal{M}}^{\max}(\mathcal{X})$. Then $(\mathcal{Y}, \mathcal{X})$ is a weak radial pair such that the pair of inclusion maps $\mathcal{Y} \hookrightarrow \mathcal{M}_G$ and $\mathcal{X} \hookrightarrow \mathcal{M}_{\mathbf{H}}$ is a morphism. In particular, for any $V \in \hat{K}_M$ we have

$$\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{Y}) = \text{Hom}_K(V, \mathcal{Y}) \cap \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G),$$

and the bijection

$$(8.10) \quad \tilde{\Gamma}_{\mathcal{M}}^V : \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{Y}) \xrightarrow{\sim} \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}).$$

If \mathcal{M} is a radial pair then so is $(\mathcal{Y}, \mathcal{X})$ by Proposition 8.12 (ii).

- (iii) The weak radial pair $(\Xi_{\mathcal{M}}^{\min}(\mathcal{X}), \mathcal{X})$ always satisfies (rad-1). If \mathcal{M} satisfies (rad-2) then $(\Xi_{\mathcal{M}}^{\min}(\mathcal{X}), \mathcal{X})$ is a radial pair. In this case, for any $V \in \hat{K}_{\text{sp}}$ we have

$$(8.11) \quad \text{Hom}_K(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{X})) = \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{X}))$$

and hence the bijection

$$\tilde{\Gamma}_{\mathcal{M}}^V : \text{Hom}_K(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{X})) \xrightarrow{\sim} \text{Hom}_W(V^M, \mathcal{X}).$$

Example 8.24. For any $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{w-rad}}$, $\Xi_{\mathcal{M}}^{\min}(\{0\}) = \{0\}$ and $\Xi_{\mathcal{M}}^{\max}(\mathcal{M}_{\mathbf{H}}) = \mathcal{M}_G$.

Example 8.25. Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then (5.1) and (5.3) imply

$$\Xi_0^{\min}(\mathcal{A}(A, \lambda)) \subset \mathcal{A}(G/K, \lambda)_{K\text{-finite}} \subset \Xi_0^{\max}(\mathcal{A}(A, \lambda)).$$

In particular $(\mathcal{A}(G/K, \lambda)_{K\text{-finite}}, \mathcal{A}(A, \lambda))$ is a radial pair.

We regard that (8.10) describes the correspondence of the multiplicity of the K -representation V in \mathcal{Y} with the multiplicity of the W -representation V_{single}^M in \mathcal{X} (cf. Remark 8.8).

Proof of Theorem 8.23. Let $\mathcal{I}_{\mathbf{H}} : \mathcal{X} \hookrightarrow \mathcal{M}_{\mathbf{H}}$ is the inclusion map. It follows from Proposition 8.20 that there exists a unique $\mathcal{I}_G : \Xi_{\text{w-rad}}(\mathcal{X}) \rightarrow \mathcal{M}_G$ such that $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}})$ is a morphism of $\mathcal{C}_{\text{w-rad}}$. From the proof of Proposition 8.20 we can see \mathcal{I}_G maps $v \otimes \varphi_V \in V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})$ ($V \in \hat{K}_M$) to $\Phi_V[v]$ where $\Phi_V \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$

is a unique element such that $\tilde{\Gamma}_{\mathcal{M}}^V(\Phi_V) = \varphi_V$. Note $\Phi_V[v]$ is written as $v \otimes \Phi_V$ if we identify the V -isotypic component of \mathcal{M}_G with $V \otimes \text{Hom}_K(V, \mathcal{M}_G)$. This shows $\text{Im } \mathcal{I}_G = \Xi_{\mathcal{M}}^{\min}(\mathcal{X})$. Hence by Proposition 8.19 and Proposition 8.12 (iii), $(\Xi_{\mathcal{M}}^{\min}(\mathcal{X}), \mathcal{X})$ is a weak radial pair satisfying (rad-1). Hence in particular for any $V \in \hat{K}_M$ we have

$$(8.12) \quad \tilde{\Gamma}_{\mathcal{M}}^V(\text{Hom}_K(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{X}))) = \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}),$$

$$(8.13) \quad \tilde{\Gamma}_{\mathcal{M}}^V(\text{Hom}_K(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{X})) \cap \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)) = \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}).$$

From (8.12) we get $\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) \subset \Xi_{\mathcal{M}}^{\max}(\mathcal{X})$.

In general, if $\{\mathcal{Y}_{\nu}\}$ is a family of $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodules of \mathcal{M}_G then

$$\text{Hom}_K\left(V, \sum_{\nu} \mathcal{Y}_{\nu}\right) = \sum_{\nu} \text{Hom}_K(V, \mathcal{Y}_{\nu}) \subset \text{Hom}_K(V, \mathcal{M}_G).$$

Hence for any $V \in \hat{K}_M$

$$(8.14) \quad \tilde{\Gamma}_{\mathcal{M}}^V(\text{Hom}_K(V, \Xi_{\mathcal{M}}^{\max}(\mathcal{X}))) \subset \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}).$$

Now suppose a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule $\mathcal{Y} \subset \mathcal{M}_G$ is such that $\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) \subset \mathcal{Y} \subset \Xi_{\mathcal{M}}^{\max}(\mathcal{X})$. Then by virtue of (8.12)–(8.14) we have

$$\begin{aligned} \tilde{\Gamma}_{\mathcal{M}}^V(\text{Hom}_K(V, \mathcal{Y})) &= \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}), \\ \tilde{\Gamma}_{\mathcal{M}}^V(\text{Hom}_K(V, \mathcal{Y}) \cap \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)) &= \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}). \end{aligned}$$

Hence $(\mathcal{Y}, \mathcal{X})$ is a subobject of $(\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{Ch}}$. Accordingly $(\mathcal{Y}, \mathcal{X}) \in \mathcal{C}_{\text{w-rad}}$ by Proposition 8.12 (ii). Similarly, if $(\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ additionally satisfies (rad-1) or (rad-2) then so does $(\mathcal{Y}, \mathcal{X})$.

What remains to be shown is (8.11) when \mathcal{M} satisfies (rad-2) and $V \in \hat{K}_{\text{sp}}$. In this case it holds that

$$\text{Hom}_K(V, P_G(F)) = \text{Hom}_K^{2 \rightarrow 2}(V, P_G(F)) \quad \text{for any } F \in \hat{K}_M$$

by Proposition 6.3 (i). Let us consider the surjective map

$$\text{Hom}_K(V, \Xi_{\text{w-rad}}(\mathcal{X})) \xrightarrow{\mathcal{I}_G \circ \cdot} \text{Hom}_K(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{X})).$$

Express any element Φ in the left-hand side as in (8.9) and take $\Phi_F^i \in \text{Hom}_K^{2 \rightarrow 2}(F, \mathcal{M}_G)$ so that $\tilde{\Gamma}_{\mathcal{M}}^F(\Phi_F^i) = \mathcal{I}_{\mathbf{H}} \circ \varphi_F^i$. Then

$$\mathcal{I}_G \circ \Phi = \sum_{F \in \hat{K}_M} \sum_{i=1}^{q_F} \Phi_F^i \circ \Psi_F^i$$

where $\Psi_F^i \in \text{Hom}_K(V, P_G(F)) = \text{Hom}_K^{2 \rightarrow 2}(V, P_G(F))$. Thus from (rad-2) for \mathcal{M} we have

$$\mathcal{I}_G \circ \Phi \in \text{Hom}_K(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{X})) \cap \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) = \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{X})).$$

This proves (8.11). \square

In general, when $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{w-rad}}$ has a radial restriction $\gamma_{\mathcal{M}}$ and \mathcal{X} is an \mathbf{H} -submodule of $\mathcal{M}_{\mathbf{H}}$, $\gamma_{\mathcal{M}}$ is not necessarily a radial restriction of $(\Xi_{\mathcal{M}}^{\min}(\mathcal{X}), \mathcal{X})$ because it may happen that $\gamma_{\mathcal{M}}(\Xi_{\mathcal{M}}^{\min}(\mathcal{X})) \not\subset \mathcal{X}$. But such a thing never happens if $\mathcal{M} = (P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$ and $\gamma_{\mathcal{M}} = \gamma$.

Proposition 8.26 (the correspondence $\Xi_{\mathcal{M}}^{\natural}$). *Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{w-rad}}$ and a linear map $\gamma_{\mathcal{M}}$ satisfies the assumption of Lemma 8.13, that is, Conditions (rest-1'), (rest-2) and (rest-3). For an \mathbf{H} -submodule \mathcal{X} of $\mathcal{M}_{\mathbf{H}}$ set*

$$\Xi_{\mathcal{M}}^{\natural}(\mathcal{X}) = \sum \{ \mathcal{V} \subset \mathcal{M}_G; \text{ a } K\text{-stable } \mathbb{C}\text{-subspace with } \gamma_{\mathcal{M}}(\mathcal{V}) \subset \mathcal{X} \}.$$

Then we have the following:

- (i) $\Xi_{\mathcal{M}}^{\natural}(\mathcal{X})$ is a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule of \mathcal{M}_G such that $\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) \subset \Xi_{\mathcal{M}}^{\natural}(\mathcal{X}) \subset \Xi_{\mathcal{M}}^{\max}(\mathcal{X})$.
- (ii) Suppose $\mathcal{Y} \subset \mathcal{M}_G$ is a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule such that $\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) \subset \mathcal{Y} \subset \Xi_{\mathcal{M}}^{\natural}(\mathcal{X})$. Then the linear map $\gamma_{\mathcal{M}}|_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{X}$ satisfies (rest-1'), (rest-2) and (rest-3) for the sub weak radial pair $(\mathcal{Y}, \mathcal{X})$. Also, a linear map $\gamma_{\mathcal{Q}} : \mathcal{M}_G/\mathcal{Y} \rightarrow \mathcal{M}_{\mathbf{H}}/\mathcal{X}$ is naturally induced from $\gamma_{\mathcal{M}}$, which satisfies (rest-1'), (rest-2) and (rest-3) for the quotient weak radial pair $\mathcal{Q} := (\mathcal{M}_G/\mathcal{Y}, \mathcal{M}_{\mathbf{H}}/\mathcal{X})$. If $\gamma_{\mathcal{M}}$ satisfies (rest-1) then so does $\gamma_{\mathcal{M}}|_{\mathcal{Y}}$.
- (iii) If $\mathcal{Y} = \Xi_{\mathcal{M}}^{\natural}(\mathcal{X})$ in (ii), then $\mathcal{Q} = (\mathcal{M}_G/\Xi_{\mathcal{M}}^{\natural}(\mathcal{X}), \mathcal{M}_{\mathbf{H}}/\mathcal{X})$ is a radial pair and $\gamma_{\mathcal{Q}} : \mathcal{M}_G/\Xi_{\mathcal{M}}^{\natural}(\mathcal{X}) \rightarrow \mathcal{M}_{\mathbf{H}}/\mathcal{X}$ is a radial restriction of \mathcal{Q} satisfying (rest-3).
- (iv) The radial restriction γ for $(P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$ satisfies the assumption of the proposition (cf. Example 8.15). In this case we use the symbol Ξ^{\natural} instead of $\Xi_{\mathcal{M}}^{\natural}$.

Proof. Suppose $\mathcal{V} \subset \mathcal{M}_G$ is K -stable and $\gamma_{\mathcal{M}}(\mathcal{V}) \subset \mathcal{X}$. Then $\mathfrak{g}_{\mathbb{C}}\mathcal{V}$ is also K -stable and $\gamma_{\mathcal{M}}(\mathfrak{g}_{\mathbb{C}}\mathcal{V}) = \gamma_{\mathcal{M}}((\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}})\mathcal{V}) = \mathfrak{a}_{\mathbb{C}}\gamma_{\mathcal{M}}(\mathcal{V}) + \gamma_{\mathcal{M}}(\mathcal{V}) \subset \mathcal{X}$ by (8.2) and (8.3). Thus $\Xi_{\mathcal{M}}^{\natural}(\mathcal{X})$ is stable under the action of $(\mathfrak{g}_{\mathbb{C}}, K)$. It is clear that $\Xi_{\mathcal{M}}^{\natural}(\mathcal{X}) \subset \Xi_{\mathcal{M}}^{\max}(\mathcal{X})$. Recall $\Xi_{\mathcal{M}}^{\min}(\mathcal{X})$ is generated by

$$v \otimes \Phi \in V \otimes \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \quad \text{with } V \in \widehat{K}_M \text{ and } \tilde{\Gamma}_{\mathcal{M}}^V(\Phi) \in \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}).$$

Since $v \otimes \Phi = \Phi[v]$ we have

$$\begin{aligned} \gamma_{\mathcal{M}}(V \otimes \Phi) &= \gamma_{\mathcal{M}}(\Phi[V]) \\ &= \gamma_{\mathcal{M}}(\Phi[V^M]) && (\because (8.4)) \\ &= \gamma_{\mathcal{M}}(\Phi[V_{\text{single}}^M]) && (\because (\text{rest-1}')) \\ &= \tilde{\Gamma}_{\mathcal{M}}^V(\Phi)[V_{\text{single}}^M] && (\because (\text{rest-2})) \\ &\subset \mathcal{X}. \end{aligned}$$

This proves $\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) \subset \Xi_{\mathcal{M}}^{\natural}(\mathcal{X})$ and hence (i).

Suppose \mathcal{Y} is as in (ii). It is immediate from Proposition 8.12 (ii) that $\gamma_{\mathcal{M}}|_{\mathcal{Y}}$ satisfies (rest-1'), (rest-2) and (rest-3) for $(\mathcal{Y}, \mathcal{X})$ (and (rest-1) if $\gamma_{\mathcal{M}}$ satisfies (rest-1)). The pair of inclusion maps $(\mathcal{Y}, \mathcal{X}) \rightarrow (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ induces a morphism $(\mathcal{J}_G, \mathcal{J}_{\mathbf{H}}) : \mathcal{M} \rightarrow (\mathcal{M}_G/\mathcal{Y}, \mathcal{M}_{\mathbf{H}}/\mathcal{X})$ of $\mathcal{C}_{\text{w-rad}}$. Since $\gamma_{\mathcal{Q}} \circ \mathcal{J}_G = \mathcal{J}_{\mathbf{H}} \circ \gamma_{\mathcal{M}}$ we can easily observe from Proposition 8.12 (iii) that $\gamma_{\mathcal{Q}}$ satisfies (rest-1'), (rest-2) and (rest-3). Thus (ii) is proved.

Finally suppose $\mathcal{Y} = \Xi_{\mathcal{M}}^{\natural}(\mathcal{X})$ and let us prove $\gamma_{\mathcal{Q}}$ satisfies (rest-1). For this purpose let $V \in \widehat{K}_M$ and take any $\Phi \in \text{Hom}_K(V, \mathcal{M}_G/\mathcal{Y})$ such that $(\gamma_{\mathcal{Q}} \circ \Phi)[V_{\text{double}}^M] = \{0\}$. Then there exists $\Phi' \in \text{Hom}_K(V, \mathcal{M}_G)$ such that $\Phi = \mathcal{J}_G \circ \Phi'$. Since $(\mathcal{J}_{\mathbf{H}} \circ \gamma_{\mathcal{M}} \circ \Phi')[V_{\text{double}}^M] = (\gamma_{\mathcal{Q}} \circ \mathcal{J}_G \circ \Phi')[V_{\text{double}}^M] = (\gamma_{\mathcal{Q}} \circ \Phi)[V_{\text{double}}^M] = \{0\}$, we have $(\gamma_{\mathcal{M}} \circ \Phi')[V_{\text{double}}^M] \subset \mathcal{X}$. Now it follows from (Ch-0) that there exists $\Phi'' \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$

such that $\tilde{\Gamma}_{\mathcal{M}}^V(\Phi'') = \tilde{\Gamma}_{\mathcal{M}}^V(\Phi')$. But then we have

$$\begin{aligned} (\gamma_{\mathcal{M}} \circ (\Phi' - \Phi''))|_{V_{\text{single}}^M} &= \tilde{\Gamma}_{\mathcal{M}}^V(\Phi' - \Phi'') = 0, & (\because (\text{rest-2})) \\ (\gamma_{\mathcal{M}} \circ (\Phi' - \Phi''))|_{V_{\text{double}}^M} &= (\gamma_{\mathcal{M}} \circ \Phi')|_{V_{\text{double}}^M} \subset \mathcal{X}, & (\because (\text{rest-1}')) \\ \therefore \gamma_{\mathcal{M}}((\Phi' - \Phi'')[V^M]) &\subset \mathcal{X}, \\ \therefore \gamma_{\mathcal{M}}((\Phi' - \Phi'')[V]) &\subset \mathcal{X}, & (\because (8.4)) \\ \therefore (\Phi' - \Phi'')[V] &\subset \Xi_{\mathcal{M}}^{\mathfrak{b}}(\mathcal{X}) = \mathcal{Y}. \end{aligned}$$

Hence $\Phi = \mathcal{J}_G \circ \Phi' = \mathcal{J}_G \circ \Phi'' \in \text{Hom}_{K^{2 \rightarrow 2}}(V, \mathcal{M}_G/\mathcal{Y})$. This shows $\gamma_{\mathcal{Q}}$ satisfies (rest-1), proving (iii). \square

In §17 the correspondence

$$\Xi^{\min} : \{\mathbf{H}\text{-submodules of } P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})\} \rightarrow \{(\mathfrak{g}_{\mathbb{C}}, K)\text{-submodules of } P_G(\mathbb{C}_{\text{triv}})\}$$

will be extended to a functor sending an \mathbf{H} -module \mathcal{X} to a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module $\Xi^{\min}(\mathcal{X})$ such that $(\Xi^{\min}(\mathcal{X}), \mathcal{X})$ is a radial pair with some canonical radial restriction satisfying (rest-3). Proposition 8.26 will play a key role in that argument. We conclude this section with the following:

Theorem 8.27. *Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$, $\mathcal{N} = (\mathcal{N}_G, \mathcal{N}_{\mathbf{H}}) \in \mathcal{C}_{\text{w-rad}}$ and $\mathcal{I} = (\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism. Then for any \mathbf{H} -submodule $\mathcal{X} \subset \mathcal{M}_{\mathbf{H}}$*

$$(8.15) \quad \Xi_{\mathcal{N}}^{\min}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})) = \mathcal{I}_G(\Xi_{\mathcal{M}}^{\min}(\mathcal{X})) \subset \mathcal{I}_G(\Xi_{\mathcal{M}}^{\max}(\mathcal{X})) \subset \Xi_{\mathcal{N}}^{\max}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})).$$

For any \mathbf{H} -submodule $\mathcal{X}' \subset \mathcal{N}_{\mathbf{H}}$

$$(8.16) \quad \Xi_{\mathcal{M}}^{\min}(\mathcal{I}_{\mathbf{H}}^{-1}(\mathcal{X}')) \subset \mathcal{I}_G^{-1}(\Xi_{\mathcal{N}}^{\min}(\mathcal{X}')) \subset \mathcal{I}_G^{-1}(\Xi_{\mathcal{N}}^{\max}(\mathcal{X}')) = \Xi_{\mathcal{M}}^{\max}(\mathcal{I}_{\mathbf{H}}^{-1}(\mathcal{X}')).$$

In particular

$$(8.17) \quad \Xi_{\mathcal{N}}^{\min}(\text{Im } \mathcal{I}_{\mathbf{H}}) \subset \text{Im } \mathcal{I}_G \subset \Xi_{\mathcal{N}}^{\max}(\text{Im } \mathcal{I}_{\mathbf{H}}),$$

$$(8.18) \quad \Xi_{\mathcal{M}}^{\min}(\text{Ker } \mathcal{I}_{\mathbf{H}}) \subset \text{Ker } \mathcal{I}_G \subset \Xi_{\mathcal{M}}^{\max}(\text{Ker } \mathcal{I}_{\mathbf{H}}).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathcal{I}_{\mathbf{H}}} & \mathcal{I}_{\mathbf{H}}(\mathcal{X}) \\ \mathcal{J}_{\mathbf{H}} \downarrow & & \downarrow \mathcal{J}'_{\mathbf{H}} \\ \mathcal{M}_{\mathbf{H}} & \xrightarrow{\mathcal{I}_{\mathbf{H}}} & \mathcal{N}_{\mathbf{H}} \end{array}$$

where $\mathcal{J}_{\mathbf{H}}$ and $\mathcal{J}'_{\mathbf{H}}$ are inclusion maps. Then by Proposition 8.20 there uniquely exist a set of $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphisms \mathcal{J}_G , \mathcal{J}'_G and $\overline{\mathcal{I}_G}$, such that the diagram

$$\begin{array}{ccc} (\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X}) & \xrightarrow{(\overline{\mathcal{I}_G}, \mathcal{I}_{\mathbf{H}})} & (\Xi_{\text{w-rad}}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})), \mathcal{I}_{\mathbf{H}}(\mathcal{X})) \\ (\mathcal{J}_G, \mathcal{J}_{\mathbf{H}}) \downarrow & & \downarrow (\mathcal{J}'_G, \mathcal{J}'_{\mathbf{H}}) \\ (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) & \xrightarrow{(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}})} & (\mathcal{N}_G, \mathcal{N}_{\mathbf{H}}) \end{array}$$

commutes in $\mathcal{C}_{\text{w-rad}}$. Here as is shown in the proof of Theorem 8.23, $\Xi_{\mathcal{M}}^{\min}(\mathcal{X}) = \text{Im } \mathcal{J}_G$ and $\Xi_{\mathcal{N}}^{\min}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})) = \text{Im } \mathcal{J}'_G$. We also note $\overline{\mathcal{I}_G}$ is surjective since $\Xi_{\text{w-rad}}$ is exact. Hence we have

$$\Xi_{\mathcal{N}}^{\min}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})) = \text{Im } \mathcal{J}'_G = \text{Im } \mathcal{J}'_G \circ \overline{\mathcal{I}_G} = \text{Im } \mathcal{I}_G \circ \mathcal{J}_G = \mathcal{I}_G(\text{Im } \mathcal{J}_G) = \mathcal{I}_G(\Xi_{\mathcal{M}}^{\min}(\mathcal{X})),$$

namely, the first equality in (8.15). Applying this to the case where $\mathcal{X} = \mathcal{I}_{\mathbf{H}}^{-1}(\mathcal{X}')$ we have

$$\mathcal{I}_G(\Xi_{\mathcal{M}}^{\min}(\mathcal{I}_{\mathbf{H}}^{-1}(\mathcal{X}'))) = \Xi_{\mathcal{N}}^{\min}(\mathcal{I}_{\mathbf{H}}(\mathcal{I}_{\mathbf{H}}^{-1}(\mathcal{X}'))) \subset \Xi_{\mathcal{N}}^{\min}(\mathcal{X}').$$

This implies the first inclusion relation of (8.16). The first inclusion in (8.15) and the second inclusion in (8.16) are obvious from Theorem 8.23 (i). Now, for a $(\mathfrak{g}_{\mathbf{C}}, K)$ -submodule \mathcal{Y} of \mathcal{M}_G

$$\begin{aligned} \mathcal{Y} \subset \mathcal{I}_G^{-1}(\Xi_{\mathcal{N}}^{\max}(\mathcal{X}')) &\iff \mathcal{I}_G(\mathcal{Y}) \subset \Xi_{\mathcal{N}}^{\max}(\mathcal{X}') \\ &\iff \forall V \in \widehat{K}_M \quad \tilde{\Gamma}_{\mathcal{N}}^V(\text{Hom}_K(V, \mathcal{I}_G(\mathcal{Y}))) \subset \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}'). \end{aligned}$$

But since

$$\begin{aligned} \tilde{\Gamma}_{\mathcal{N}}^V(\text{Hom}_K(V, \mathcal{I}_G(\mathcal{Y}))) &= (\tilde{\Gamma}_{\mathcal{N}}^V \circ (\mathcal{I}_G \circ \cdot))(\text{Hom}_K(V, \mathcal{Y})) \\ &= ((\mathcal{I}_{\mathbf{H}} \circ \cdot) \circ \tilde{\Gamma}_{\mathcal{M}}^V)(\text{Hom}_K(V, \mathcal{Y})), \end{aligned}$$

the above condition is still equivalent to

$$\forall V \in \widehat{K}_M \quad \tilde{\Gamma}_{\mathcal{M}}^V(\text{Hom}_K(V, \mathcal{Y})) \subset \text{Hom}_W(V_{\text{single}}^M, \mathcal{I}_{\mathbf{H}}^{-1}(\mathcal{X}')) \iff \mathcal{Y} \subset \Xi_{\mathcal{M}}^{\max}(\mathcal{I}_{\mathbf{H}}^{-1}(\mathcal{X}')).$$

Thus we get the last equality in (8.16). Applying this to the case where $\mathcal{X}' = \mathcal{I}_{\mathbf{H}}(\mathcal{X})$ we get

$$\mathcal{I}_G(\Xi_{\mathcal{M}}^{\max}(\mathcal{X})) \subset \mathcal{I}_G(\Xi_{\mathcal{M}}^{\max}(\mathcal{I}_{\mathbf{H}}^{-1}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})))) = \mathcal{I}_G(\mathcal{I}_G^{-1}(\Xi_{\mathcal{N}}^{\max}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})))) \subset \Xi_{\mathcal{N}}^{\max}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})),$$

namely the last inclusion relation in (8.15).

Finally, (8.17) and (8.18) follow from (8.15), (8.16) and Example 8.24. \square

9. SPHERICAL PRINCIPAL SERIES

In this section we review the spherical principal series representation $B_G(\lambda)$ for G and the corresponding standard representation $B_{\mathbf{H}}(\lambda)$ for \mathbf{H} . For the latter we employ an unusual realization so that a certain “restriction map” $\gamma_{B(\lambda)} : B_G(\lambda) \rightarrow B_{\mathbf{H}}(\lambda)$ can easily be defined. It will turn out that $B(\lambda) := (B_G(\lambda)_{K\text{-finite}}, B_{\mathbf{H}}(\lambda))$ is a radial pair with radial restriction $\gamma_{B(\lambda)}$. We also review standard invariant sesquilinear forms for principal series representations, which play important roles in later sections.

We discuss these things in a slightly more general setting for the sake of application in §17. Suppose a finite-dimensional $\mathfrak{a}_{\mathbf{C}}$ -module (σ, \mathcal{U}) is given. The action σ of \mathfrak{a} on \mathcal{U} can be integrated to the action of the simply connected Lie group A :

$$A \ni a \longmapsto a^{\sigma} := \exp \sigma(\log a) \in \text{End}_{\mathbf{C}} \mathcal{U}.$$

We denote by \mathcal{U}^* the linear space of antilinear functionals on \mathcal{U} . Let $(\cdot, \cdot)_{\mathcal{U}}$ denote the canonical sesquilinear form on $\mathcal{U}^* \times \mathcal{U}$. Then \mathcal{U}^* is naturally an $\mathfrak{a}_{\mathbf{C}}$ -module by

$$(\sigma^*(\xi)u^*, u)_{\mathcal{U}} = -(u^*, \sigma(\xi)u)_{\mathcal{U}} \quad \text{for } \xi \in \mathfrak{a}, u^* \in \mathcal{U}^* \text{ and } u \in \mathcal{U}.$$

Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and let \mathbb{C}_{λ} be the \mathbb{C} endowed with the $\mathfrak{a}_{\mathbb{C}}$ -module structure by $\mathfrak{a}_{\mathbb{C}} \ni \xi \mapsto \lambda(\xi) \in \text{End}_{\mathbb{C}} \mathbb{C}_{\lambda}$. We naturally identify $\mathbb{C}_{-\lambda}^*$ with $\mathbb{C}_{\bar{\lambda}}$.

Definition 9.1. We put

$$\begin{aligned} \text{Ind}_{MAN}^G(\mathcal{U}) &= \left\{ F : G \xrightarrow{C^\infty} \mathcal{U}; \quad F(gman) = a^{-\sigma-\rho} F(g) \right. \\ &\quad \left. \text{for } (g, m, a, n) \in G \times M \times A \times N \right\}, \\ B_G(\lambda) &= \text{Ind}_{MAN}^G(\mathbb{C}_{-\lambda}) \\ &= \left\{ F \in C^\infty(G); \quad F(gman) = a^{\lambda-\rho} F(g) \right. \\ &\quad \left. \text{for } (g, m, a, n) \in G \times M \times A \times N \right\}. \end{aligned}$$

We consider $\text{Ind}_{MAN}^G(\mathcal{U})$ as a G -module by the similar action to $\ell(\cdot)$. Furthermore we define a sesquilinear form ${}_{\mathcal{U}}(\cdot, \cdot)_{\mathcal{U}^*}^G$ on $\text{Ind}_{MAN}^G(\mathcal{U}) \times \text{Ind}_{MAN}^G(\mathcal{U}^*)$ by

$${}_{\mathcal{U}}(F_1, F_2)_{\mathcal{U}^*}^G = \int_K (F_1(k), F_2(k))_{\mathcal{U}^*} dk.$$

Here dk is the Haar measure on K with $\int dk = 1$. In particular the sesquilinear form ${}_{\lambda}(\cdot, \cdot)_{-\bar{\lambda}}^G$ on $B_G(\lambda) \times B_G(-\bar{\lambda})$ is defined by

$${}_{\lambda}(F_1, F_2)_{-\bar{\lambda}}^G = \int_K F_1(k) \overline{F_2(k)} dk.$$

Proposition 9.2. *The sesquilinear form ${}_{\mathcal{U}}(\cdot, \cdot)_{\mathcal{U}^*}^G$ is invariant and non-degenerate.*

Proof. Use the G -invariance of the linear functional

$$B_G(-\rho) \ni F(g) \mapsto \int_K F(k) dk \in \mathbb{C}$$

(cf. [Hel4, Ch. I, Lemma 5.19]). □

Definition 9.3. Let w_0 be as in Definition 7.1 and let ${}^t \cdot : \mathbf{H} \rightarrow \mathbf{H}$ be a unique algebra anti-automorphism such that

$$\begin{cases} {}^t w = w^{-1} & \text{for } w \in W, \\ {}^t \xi = -w_0 w_0(\xi) w_0 & \text{for } \xi \in \mathfrak{a}_{\mathbb{C}}. \end{cases}$$

We put

$$\begin{aligned} \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U}) &= \{F \in \text{Hom}_{\mathbb{C}}(\mathbf{H}, \mathcal{U}); F(h {}^t \xi) = \sigma(\xi) F(h) \text{ for } h \in \mathbf{H} \text{ and } \xi \in \mathfrak{a}_{\mathbb{C}}\}, \\ B_{\mathbf{H}}(\lambda) &= \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathbb{C}_{-\lambda}) \\ &= \{F \in \mathbf{H}^*; F(h {}^t \xi) = -\lambda(\xi) F(h) \text{ for } h \in \mathbf{H} \text{ and } \xi \in \mathfrak{a}_{\mathbb{C}}\}. \end{aligned}$$

We consider $\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U})$ as a left \mathbf{H} -module by $hF(\cdot) = F({}^t h \cdot)$. Furthermore we define a sesquilinear form ${}_{\mathcal{U}}(\cdot, \cdot)_{\mathcal{U}^*}^{\mathbf{H}}$ on $\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U}) \times \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U}^*)$ by

$${}_{\mathcal{U}}(F_1, F_2)_{\mathcal{U}^*}^{\mathbf{H}} = \frac{1}{|W|} \sum_{w \in W} (F_1(w), F_2(w))_{\mathcal{U}^*}.$$

In particular the sesquilinear form ${}_{\lambda}(\cdot, \cdot)_{-\bar{\lambda}}^{\mathbf{H}}$ on $B_{\mathbf{H}}(\lambda) \times B_{\mathbf{H}}(-\bar{\lambda})$ is defined by

$${}_{\lambda}(F_1, F_2)_{-\bar{\lambda}}^{\mathbf{H}} = \frac{1}{|W|} \sum_{w \in W} F_1(w) \overline{F_2(w)}.$$

Definition 9.4. let $\cdot^* : \mathbf{H} \rightarrow \mathbf{H}$ be a unique antilinear anti-automorphism such that

$$\begin{cases} w^* = w^{-1} & \text{for } w \in W, \\ \xi^* = -w_0 w_0(\xi) w_0 & \text{for } \xi \in \mathfrak{a}. \end{cases}$$

Suppose \mathcal{X}_1 and \mathcal{X}_2 are \mathbf{H} -modules. Then a sesquilinear form $(\cdot, \cdot) : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathbb{C}$ is called invariant if it satisfies

$$(hx_1, x_2) = (x_1, h^* x_2) \quad \text{for any } x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2 \text{ and } h \in \mathbf{H}.$$

Proposition 9.5. *The sesquilinear form ${}_{\mathcal{U}}(\cdot, \cdot)_{\mathcal{U}^*}^{\mathbf{H}}$ is invariant and non-degenerate.*

Proof. Since $\mathbf{H} = {}^t\mathbf{H} = {}^t(S(\mathfrak{a}_{\mathbb{C}}) \otimes \mathbb{C}W) = \mathbb{C}W \otimes {}^tS(\mathfrak{a}_{\mathbb{C}})$, restriction to $\mathbb{C}W$ gives the W -isomorphism $\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U}) \xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(\mathbb{C}W, \mathcal{U})$. Hence the non-degeneracy and the W -invariance are clear. In general, for $\xi \in \mathfrak{a}_{\mathbb{C}}$ and $w \in W$ it holds that

$$(9.1) \quad w \xi w^{-1} = w(\xi) + \sum_{\alpha \in R_1^+ \cap ww_0 R_1^+} \mathbf{m}_1(\alpha)(w^{-1}\alpha)(\xi) s_{\alpha}$$

in \mathbf{H} (cf. [Op1, Proposition 1.1 (1)]). We also note $\xi w_0 = -w_0 {}^t(w_0(\xi))$ for $\xi \in \mathfrak{a}_{\mathbb{C}}$. Hence for any $\xi \in \mathfrak{a}_{\mathbb{C}}$

$$\begin{aligned} & {}_{\mathcal{U}}(\xi F_1, F_2)_{\mathcal{U}^*}^{\mathbf{H}} \\ &= \frac{1}{|W|} \sum_{w \in W} (F_1(-w_0 w_0(\xi) w_0 w), F_2(w))_{\mathcal{U}^*} \\ &= \frac{1}{|W|} \sum_{w \in W} (F_1(-w^{-1} w w_0 w_0(\xi) w_0 w^{-1} w_0), F_2(w^{-1} w_0))_{\mathcal{U}^*} \\ &= \frac{1}{|W|} \sum_{w \in W} (F_1(w^{-1}(-w(\xi) - \sum_{\alpha \in R_1^+ \cap w R_1^+} \mathbf{m}_1(\alpha)(w^{-1}\alpha)(\xi) s_{\alpha}) w_0), F_2(w^{-1} w_0))_{\mathcal{U}^*} \\ &= \frac{1}{|W|} \sum_{w \in W} (\sigma(w_0 w(\xi)) F_1(w^{-1} w_0), F_2(w^{-1} w_0))_{\mathcal{U}^*} \\ &\quad - \frac{1}{|W|} \sum_{\alpha \in R_1^+} \sum_{\substack{w \in W; \\ w^{-1}\alpha \in R^+}} \mathbf{m}_1(\alpha)(w^{-1}\alpha)(\xi) (F_1(w^{-1} s_{\alpha} w_0), F_2(w^{-1} w_0))_{\mathcal{U}^*} \\ &= \frac{1}{|W|} \sum_{w \in W} (F_1(w^{-1} w_0), \sigma^*(-w_0 w(\bar{\xi})) F_2(w^{-1} w_0))_{\mathcal{U}^*} \\ &\quad - \frac{1}{|W|} \sum_{\alpha \in R_1^+} \sum_{\substack{w \in W; \\ w^{-1}\alpha \in w_0 R^+}} \mathbf{m}_1(\alpha)((s_{\alpha} w)^{-1}\alpha)(\xi) (F_1((s_{\alpha} w)^{-1} s_{\alpha} w_0), F_2((s_{\alpha} w)^{-1} w_0))_{\mathcal{U}^*} \\ &= \frac{1}{|W|} \sum_{w \in W} (F_1(w^{-1} w_0), F_2(w^{-1} w(\bar{\xi}) w_0))_{\mathcal{U}^*} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{|W|} \sum_{\alpha \in R_1^+} \sum_{\substack{w \in W; \\ w^{-1}\alpha \in w_0 R^+}} \mathbf{m}_1(\alpha)(w^{-1}\alpha)(\xi) (F_1(w^{-1}w_0), F_2(w^{-1}s_\alpha w_0))_{\mathcal{U}^*} \\
& = \frac{1}{|W|} \sum_{w \in W} (F_1(w^{-1}w_0), F_2(w^{-1}(w(\bar{\xi}) + \sum_{\alpha \in R_1^+ \cap ww_0 R_1^+} \mathbf{m}_1(\alpha)(w^{-1}\alpha)(\bar{\xi})s_\alpha)w_0))_{\mathcal{U}^*} \\
& = \frac{1}{|W|} \sum_{w \in W} (F_1(w^{-1}w_0), F_2(\bar{\xi} w^{-1}w_0))_{\mathcal{U}^*} \\
& = {}_{\mathcal{U}}(F_1, \xi^* F_2)_{\mathcal{U}^*}^{\mathbf{H}}. \quad \square
\end{aligned}$$

Remark 9.6. This proof is essentially the same as Opdam's proof of [Op1, Theorem 4.2 (1)]. Indeed $B_{\mathbf{H}}(-w_0\lambda)$ is isomorphic to I_λ in [Op1] and ${}_{-w_0\lambda}(\cdot, \cdot)_{w_0\bar{\lambda}}^{\mathbf{H}}$ equals the sesquilinear form (\cdot, \cdot) of [Op1, Definition 7.2] up to a scalar multiple.

Definition 9.7. Define the linear map $\gamma_{\text{Ind}(\mathcal{U})} : \text{Ind}_{MAN}^G(\mathcal{U}) \rightarrow \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U})$ by

$$\text{Ind}_{MAN}^G(\mathcal{U}) \ni F(g) \mapsto (W \ni w \mapsto F(\bar{w}) \in \mathcal{U}) \in \text{Hom}_{\mathbb{C}}(\mathbb{C}W, \mathcal{U}) \simeq \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U}).$$

Here for $w \in W$, \bar{w} is any lift of w in $N_K(\mathfrak{a})$. If $\mathcal{U} = \mathbb{C}_{-\lambda}$ then the corresponding map $\gamma_{B(\lambda)} : B_G(\lambda) \rightarrow B_{\mathbf{H}}(\lambda)$ is defined by

$$B_G(\lambda) \ni F(g) \mapsto (W \ni w \mapsto F(\bar{w}) \in \mathbb{C}) \in (\mathbb{C}W)^* \simeq B_{\mathbf{H}}(\lambda).$$

Theorem 9.8. $\text{Ind}(\mathcal{U}) := (\text{Ind}_{MAN}^G(\mathcal{U})_{K\text{-finite}}, \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U}))$ is a radial pair with radial restriction $\gamma_{\text{Ind}(\mathcal{U})}$. Moreover, for any $V \in \widehat{K}_M$, $\Gamma_{\text{Ind}(\mathcal{U})}^V$ defined by (8.1) gives a linear bijection

$$(9.2) \quad \Gamma_{\text{Ind}(\mathcal{U})}^V : \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{U})) \xrightarrow{\sim} \text{Hom}_W(V^M, \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U})).$$

In particular $B(\lambda) := (B_G(\lambda)_{K\text{-finite}}, B_{\mathbf{H}}(\lambda))$ is a radial pair with radial restriction $\gamma_{B(\lambda)}$.

Proof. Suppose $V \in \widehat{K}$. Let $(V^M)^\perp$ be the orthogonal complement of V^M in V with respect to a K -invariant inner product and $p^V : V \rightarrow V^M$ the orthogonal projection. If $\Phi \in \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{U}))$ then $\Phi[v](1) = \Phi[p^V(v)](1)$ for $v \in V$ since the linear functional $V \ni v \mapsto \Phi[v](1) \in \mathbb{C}$ is M -invariant. Hence

$$(9.3) \quad \Phi[v](kan) = a^{-\sigma-\rho} \Phi[p^V(k^{-1}v)](1) \quad \text{for } v \in V \text{ and } (k, a, n) \in K \times A \times N.$$

From this one sees all the K -types of $\Gamma_{\text{Ind}(\mathcal{U})}^V$ belong to \widehat{K}_M . So we assume $V \in \widehat{K}_M$. Now clearly $\Gamma_{\text{Ind}(\mathcal{U})}^V$ maps Hom_K into Hom_W :

$$\Gamma_{\text{Ind}(\mathcal{U})}^V : \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{U})) \longrightarrow \text{Hom}_W(V^M, \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U})).$$

We assert this is a bijection. Indeed, the injectivity follows from (9.3). In addition, if $\varphi \in \text{Hom}_W(V^M, \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U}))$ then

$$\Phi : V \ni v \mapsto (G \ni g = kan \mapsto a^{-\sigma-\rho} \varphi[p^V(k^{-1}v)](1) \in \mathcal{U}) \in \text{Ind}_{MAN}^G(\mathcal{U})$$

defines an element of $\text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{U}))$ such that $\varphi = \Gamma_{\text{Ind}(\mathcal{U})}^V(\Phi)$. We thus get (9.2). Hence if we define $\text{Hom}_K^{2 \rightarrow 2}(V, \text{Ind}_{MAN}^G(\mathcal{U}))$ and $\tilde{\Gamma}_{\text{Ind}(\mathcal{U})}^V$ so that (rest-1) and

(rest-2) are valid, then (Ch-0) holds and $\text{Ind}(\mathcal{U}) \in \mathcal{C}_{\text{Ch}}$. Note that (9.2) induces a bijection

$$\Gamma_{\text{Ind}(\mathcal{U})} : \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \text{Ind}_{MAN}^G(\mathcal{U})_{K\text{-finite}}) \xrightarrow{\sim} \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V^M), \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U})).$$

In order to prove (w-rad), (rad-1) and (rad-2), suppose $\Phi \in \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{U})) \simeq \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \text{Ind}_{MAN}^G(\mathcal{U})_{K\text{-finite}})$, $E \in \widehat{K}_M$, and $\Psi \in \text{Hom}_K(E, P_G(V))$. Take a basis $\{v_1, \dots, v_m, \dots, v_n\}$ of V so that $\{v_1, \dots, v_m\}$ and $\{v_{m+1}, \dots, v_n\}$ are respectively bases of V^M and $(V^M)^\perp$. If for any $e \in E^M$ we write

$$\Psi[e] = \sum_{i=1}^n D_i \otimes v_i \text{ with } D_i \in U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}}),$$

then we have

$$\Gamma_V^E(\Psi)[e] = \sum_{i=1}^m \gamma(D_i) \otimes v_i \text{ with } \gamma(D_i) \in S(\mathfrak{a}_{\mathbb{C}}),$$

and

$$\begin{aligned} (\Phi \circ \Psi)[e](1) &= \sum_{i=1}^n (D_i \Phi[v_i])(1) = \sum_{i=1}^n \sigma(\gamma(D_i))(\Phi[v_i](1)) \\ &= \sum_{i=1}^m \sigma(\gamma(D_i))(\Phi[v_i](1)) = \sum_{i=1}^m \sigma(\gamma(D_i))(\Gamma_{\text{Ind}(\mathcal{U})}^V(\Phi)[v_i](1)) \\ &= \sum_{i=1}^m (\gamma(D_i) \Gamma_{\text{Ind}(\mathcal{U})}^V(\Phi)[v_i])(1) = (\Gamma_{\text{Ind}(\mathcal{U})}(\Phi) \circ \Gamma_V^E(\Psi))[e](1). \end{aligned}$$

Now since $\Gamma_{\text{Ind}(\mathcal{U})}^E(\Phi \circ \Psi) \in \text{Hom}_W(E^M, \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{U}))$, for any $w \in W$

$$\begin{aligned} (9.4) \quad \Gamma_{\text{Ind}(\mathcal{U})}^E(\Phi \circ \Psi)[e](w) &= \Gamma_{\text{Ind}(\mathcal{U})}^E(\Phi \circ \Psi)[w^{-1}e](1) = (\Phi \circ \Psi)[\bar{w}^{-1}e](1) \\ &= (\Gamma_{\text{Ind}(\mathcal{U})}(\Phi) \circ \Gamma_V^E(\Psi))[w^{-1}e](1). \end{aligned}$$

If $\Phi \in \text{Hom}^{2 \rightarrow 2}$, $\Psi \in \text{Hom}^{2 \rightarrow 2}$, $e \in E_{\text{double}}^M$ and $w \in W$ then

$$\begin{aligned} (\Gamma_{\text{Ind}(\mathcal{U})}(\Phi) \circ \Gamma_V^E(\Psi))[w^{-1}e] &\subset \Gamma_{\text{Ind}(\mathcal{U})}(\Phi)[\Gamma_V^E(\Psi)[E_{\text{double}}^M]] \\ &\subset \Gamma_{\text{Ind}(\mathcal{U})}(\Phi)[P_{\mathbf{H}}(V_{\text{double}}^M)] = \{0\} \end{aligned}$$

and hence $\Gamma_{\text{Ind}(\mathcal{U})}^E(\Phi \circ \Psi)[e](w) = 0$. This shows (rad-2). Finally, if $\Phi \in \text{Hom}^{2 \rightarrow 2}$ or $\Psi \in \text{Hom}^{1 \rightarrow 1}$, then $(\Gamma_{\text{Ind}(\mathcal{U})}(\Phi) \circ \Gamma_V^E(\Psi))|_{E_{\text{single}}^M} = \tilde{\Gamma}_{\text{Ind}(\mathcal{U})}(\Phi) \circ \tilde{\Gamma}_V^E(\Psi)$ is a W -homomorphism. Hence in this case for each $e \in E_{\text{single}}^M$, (9.4) reduces to

$$\tilde{\Gamma}_{\text{Ind}(\mathcal{U})}^E(\Phi \circ \Psi)[e](w) = (\tilde{\Gamma}_{\text{Ind}(\mathcal{U})}(\Phi) \circ \tilde{\Gamma}_V^E(\Psi))[w^{-1}e](1) = (\tilde{\Gamma}_{\text{Ind}(\mathcal{U})}(\Phi) \circ \tilde{\Gamma}_V^E(\Psi))[e](w),$$

proving (w-rad) and (rad-1). \square

In general, for $V \in \widehat{K}_M$ we denote by V^* the linear space of antilinear functionals on V . This has a natural K -module structure isomorphic to V and the map $\bar{\cdot} : V^* \ni v^* \mapsto \overline{v^*} = \overline{\langle \cdot, v^* \rangle} \in V^*$ is a K -antilinear isomorphism. The inverse of $\bar{\cdot}$ is also denoted by $\bar{\cdot}$.

Definition 9.9. Suppose $\mathcal{M}^1 = (\mathcal{M}_G^1, \mathcal{M}_H^1)$, $\mathcal{M}^2 = (\mathcal{M}_G^2, \mathcal{M}_H^2) \in \mathcal{C}_{\text{Ch}}$. A pair of two invariant sesquilinear forms, $(\cdot, \cdot)^G$ on $\mathcal{M}_G^1 \times \mathcal{M}_G^2$ and $(\cdot, \cdot)^H$ on $\mathcal{M}_H^1 \times \mathcal{M}_H^2$, is said to be *compatible with restriction* if it satisfies the following condition:

For any $V \in \widehat{K}_{\text{qsp}}$ take a basis $\{v_1, \dots, v_{m'}, \dots, v_n\}$ of V so that $\{v_1, \dots, v_{m'}\}$ is a basis of V_{single}^M and $v_{m'+1}, \dots, v_n$ are orthogonal to V_{single}^M with respect to a K -invariant inner product of V ; Let $\{v_1^*, \dots, v_n^*\} \subset V^*$ be the dual basis of $\{v_1, \dots, v_n\}$ and put $v_i^* = \overline{v_i^*}$ for $i = 1, \dots, n$; Then for any $(\Phi_1, \Phi_2) \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G^1) \times \text{Hom}_K(V^*, \mathcal{M}_G^2) \cup \text{Hom}_K(V, \mathcal{M}_G^1) \times \text{Hom}_K^{2 \rightarrow 2}(V^*, \mathcal{M}_G^2)$

$$\sum_{i=1}^n (\Phi_1[v_i], \Phi_2[v_i^*])^G = \sum_{i=1}^{m'} (\tilde{\Gamma}_{\mathcal{M}^1}^V(\Phi_1)[v_i], \tilde{\Gamma}_{\mathcal{M}^2}^{V^*}(\Phi_2)[v_i^*])^H.$$

Proposition 9.10. Suppose $V \in \widehat{K}_M$ and take a basis $\{v_1, \dots, v_m, \dots, v_n\}$ of V so that $\{v_1, \dots, v_m\}$ is a basis of V^M and v_{m+1}, \dots, v_n are orthogonal to V^M . Define $\{v_i^*\} \subset V^*$ as in Definition 9.9. Then for any $\Phi_1 \in \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{U}))$ and $\Phi_2 \in \text{Hom}_K(V^*, \text{Ind}_{MAN}^G(\mathcal{U}^*))$

$$\sum_{i=1}^n {}_{\mathcal{U}}(\Phi_1[v_i], \Phi_2[v_i^*])_{\mathcal{U}^*}^G = \sum_{i=1}^m {}_{\mathcal{U}}(\Gamma_{\text{Ind}(\mathcal{U})}^V(\Phi_1)[v_i], \Gamma_{\text{Ind}(\mathcal{U}^*)}^{V^*}(\Phi_2)[v_i^*])_{\mathcal{U}^*}^H.$$

In particular, the pair of ${}_{\mathcal{U}}(\cdot, \cdot)_{\mathcal{U}^*}^G$ and ${}_{\mathcal{U}}(\cdot, \cdot)_{\mathcal{U}^*}^H$ is compatible with restriction.

Proof. Since $\sum_{i=1}^n v_i \otimes v_i^* = \sum_{i=1}^n k^{-1}v_i \otimes k^{-1}v_i^*$ for any $k \in K$, we calculate

$$\begin{aligned} \sum_{i=1}^n {}_{\mathcal{U}}(\Phi_1[v_i], \Phi_2[v_i^*])_{\mathcal{U}^*}^G &= \sum_{i=1}^n \int_K (\Phi_1[v_i](k), \Phi_2[v_i^*](k))_{\mathcal{U}^*} dk \\ &= \sum_{i=1}^n \int_K (\Phi_1[k^{-1}v_i](1), \Phi_2[\overline{k^{-1}v_i^*}](1))_{\mathcal{U}^*} dk \\ &= \sum_{i=1}^n \int_K (\Phi_1[v_i](1), \Phi_2[\overline{v_i^*}](1))_{\mathcal{U}^*} dk \\ &= \sum_{i=1}^n (\Phi_1[v_i](1), \Phi_2[\overline{v_i^*}](1))_{\mathcal{U}^*}. \end{aligned}$$

But since $\Phi_1[v_i](1) = 0$ for $i > m$ (cf. the proof of Theorem 9.8) and since $\sum_{i=1}^m v_i \otimes v_i^* = \sum_{i=1}^m w^{-1}v_i \otimes w^{-1}v_i^*$ for any $w \in W$, the last expression equals

$$\begin{aligned} &\sum_{i=1}^m (\Phi_1[v_i](1), \Phi_2[\overline{v_i^*}](1))_{\mathcal{U}^*} \\ &= \sum_{i=1}^m (\Gamma_{\text{Ind}(\mathcal{U})}^V(\Phi_1)[v_i](1), \Gamma_{\text{Ind}(\mathcal{U}^*)}^{V^*}(\Phi_2)[\overline{v_i^*}](1))_{\mathcal{U}^*} \\ &= \frac{1}{|W|} \sum_{w \in W} \sum_{i=1}^m (\Gamma_{\text{Ind}(\mathcal{U})}^V(\Phi_1)[w^{-1}v_i](1), \Gamma_{\text{Ind}(\mathcal{U}^*)}^{V^*}(\Phi_2)[\overline{w^{-1}v_i^*}](1))_{\mathcal{U}^*} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|W|} \sum_{w \in W} \sum_{i=1}^m (\Gamma_{\text{Ind}(\mathcal{U})}^V(\Phi_1)[v_i](w), \Gamma_{\text{Ind}(\mathcal{U}^*)}^{V^*}(\Phi_2)[v_i^*](w))_{\mathcal{U}^*} \\
&= \sum_{i=1}^m \mathcal{U} (\Gamma_{\text{Ind}(\mathcal{U})}^V(\Phi_1)[v_i], \Gamma_{\text{Ind}(\mathcal{U}^*)}^{V^*}(\Phi_2)[v_i^*])_{\mathcal{U}^*}^{\mathbf{H}}.
\end{aligned}$$

The last assertion of the proposition is easy from this. (Note $V_{\text{single}}^M \perp V_{\text{double}}^M$ [O, Lemma 3.3].) \square

10. STAR OPERATIONS ON MORPHISMS

In this section we introduce two antilinear operations

$$\begin{aligned}
\cdot^* : \text{Hom}_K(E, P_G(V)) &\longrightarrow \text{Hom}_K(V^*, P_G(E^*)), \\
\cdot^* : \text{Hom}_{\mathbb{C}}(E^M, P_{\mathbf{H}}(V^M)) &\longrightarrow \text{Hom}_{\mathbb{C}}((V^*)^M, P_{\mathbf{H}}((E^*)^M))
\end{aligned}$$

for $E, V \in \widehat{K}_M$ and study their properties. These tools will be used to prove Theorem 6.5 (iii) and other results in later sections.

Lemma 10.1. *Suppose $E, V \in \widehat{K}_M$. Then naturally*

$$\begin{aligned}
\text{Hom}_K(E, P_G(V)) &\simeq (E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K \\
&\simeq (E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K \\
&\simeq (E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes V)^K \\
&\simeq \text{Hom}_K(V^*, E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}})).
\end{aligned}$$

Here we consider $D \in U(\mathfrak{k}_{\mathbb{C}})$ acts on $e^* \in E^*$ from the right by $e^* D = {}^t D e^*$. (Recall ${}^t \cdot : U(\mathfrak{g}_{\mathbb{C}}) \rightarrow U(\mathfrak{g}_{\mathbb{C}})$ is a unique anti-automorphism such that ${}^t X = -X$ for $X \in \mathfrak{g}_{\mathbb{C}}$.)

Proof. Note that K acts on $E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V$ and $E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V$ diagonally. Since these actions are locally finite, we have the following projections to the trivial isotypic components:

$$\begin{aligned}
p_1 : E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V &\longrightarrow (E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K, \\
p_2 : E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V &\longrightarrow (E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K.
\end{aligned}$$

Thus if

$$\iota_1 : (E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K \longrightarrow E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V$$

is the inclusion map and if

$$\pi : E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V \longrightarrow E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V$$

is the canonical surjection, then $p_2 \circ \pi \circ \iota_1$ is a surjection and $p_1 \circ \iota_1 = \text{id}$. One easily checks p_1 induces a K -homomorphism

$$\tilde{p}_1 : E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V \longrightarrow (E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K$$

such that $\tilde{p}_1 \circ \pi = p_1$. Since \tilde{p}_1 factors through p_2 there exists

$$t : (E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K \longrightarrow (E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K$$

such that $t \circ p_2 = \tilde{p}_1$. Now since $t \circ (p_2 \circ \pi \circ \iota_1) = (t \circ p_2) \circ \pi \circ \iota_1 = (\tilde{p}_1 \circ \pi) \circ \iota_1 = p_1 \circ \iota_1 = \text{id}$,

$$p_2 \circ \pi \circ \iota_1 : (E^* \otimes U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K \longrightarrow (E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K$$

is a bijection. Likewise $(E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)^K \simeq (E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \otimes V)^K$ and the other isomorphisms are obvious. \square

Definition 10.2 (the star operation). Let $\cdot^* : U(\mathfrak{g}_{\mathbb{C}}) \rightarrow U(\mathfrak{g}_{\mathbb{C}})$ be a unique antilinear anti-automorphism such that $X^* = -X$ for $X \in \mathfrak{g}$ and let $\sigma : E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \rightarrow U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} E^*$ be the K -antilinear isomorphism defined by $e^* \otimes D \mapsto D^* \otimes \bar{e}^*$. For $\Psi \in \text{Hom}_K(E, P_G(V))$ let ${}^t\Psi \in \text{Hom}_K(V^*, E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}))$ be the corresponding element by Lemma 10.1. Then we define $\Psi^* \in \text{Hom}_K(V^*, P_G(E^*))$ by the following the composition:

$$\Psi^* : V^* \xrightarrow{\bar{\cdot}} V^* \xrightarrow{{}^t\Psi} E^* \otimes_{U(\mathfrak{k}_{\mathbb{C}})} U(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\sigma} U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} E^* = P_G(E^*).$$

The next proposition is easy from the definition:

Proposition 10.3. Suppose an $\text{Ad}(K)$ -stable subspace \mathbf{S} of $U(\mathfrak{g}_{\mathbb{C}})$ satisfies $\mathbf{S} \otimes U(\mathfrak{k}_{\mathbb{C}}) \xrightarrow{\sim} U(\mathfrak{g}_{\mathbb{C}})$ by multiplication (or equivalently $U(\mathfrak{k}_{\mathbb{C}}) \otimes \mathbf{S} \xrightarrow{\sim} U(\mathfrak{g}_{\mathbb{C}})$). Let $\{e_1, \dots, e_\nu\}$ and $\{v_1, \dots, v_n\}$ be bases of E and V . Let $\{e_1^*, \dots, e_\nu^*\} \subset E^*$ be the dual basis of $\{e_1, \dots, e_\nu\}$ and put $e_i^* = \bar{e}_i^*$ ($i = 1, \dots, \nu$). Define $\{v_1^*, \dots, v_n^*\} \subset V^*$ similarly. If for a given $\Psi \in \text{Hom}_K(E, P_G(V))$ we take $S_{ij} \in \mathbf{S}$ ($1 \leq i \leq \nu, 1 \leq j \leq n$) so that

$$\Psi[e_i] = \sum_{j=1}^n S_{ij} \otimes v_j \quad \text{for } i = 1, \dots, \nu,$$

then

$$\Psi^*[v_j^*] = \sum_{i=1}^\nu S_{ij}^* \otimes e_i^* \quad \text{for } j = 1, \dots, n.$$

Hence $\Psi^{**} = \Psi$.

Remark 10.4. (i) As an \mathbf{S} we can take the image of the symmetrization map for $S(\mathfrak{s}_{\mathbb{C}})$ (Recall \mathfrak{s} is the vector part of the Cartan decomposition of \mathfrak{g} .)

(ii) We cannot use $U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})$ as an \mathbf{S} since it is not $\text{Ad}(K)$ -stable.

Corollary 10.5. Suppose \mathcal{Y}_1 and \mathcal{Y}_2 are $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules and (\cdot, \cdot) is an invariant sesquilinear form on $\mathcal{Y}_1 \times \mathcal{Y}_2$. Suppose bases $\{e_i\}, \{e_i^*\}, \{v_j\}$ and $\{v_j^*\}$ are as in Proposition 10.3. For $\Psi \in \text{Hom}_K(E, P_G(V))$, $\Phi_1 \in \text{Hom}_K(V, \mathcal{Y}_1) \simeq \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V), \mathcal{Y}_1)$ and $\Phi_2 \in \text{Hom}_K(E^*, \mathcal{Y}_2) \simeq \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E^*), \mathcal{Y}_2)$ it holds that

$$\sum_{i=1}^\nu (\Phi_1 \circ \Psi[e_i], \Phi_2[e_i^*]) = \sum_{j=1}^n (\Phi_1[v_j], \Phi_2 \circ \Psi^*[v_j^*]).$$

Proof. Using the expressions of Ψ and Ψ^* in Proposition 10.3.

$$\text{L.H.S.} = \sum_{i=1}^\nu \left(\sum_{j=1}^n S_{ij} \Phi_1[v_j], \Phi_2[e_i^*] \right) = \sum_{j=1}^n \left(\Phi_1[v_j], \sum_{i=1}^\nu S_{ij}^* \Phi_2[e_i^*] \right) = \text{R.H.S.} \quad \square$$

Corollary 10.6. *Let $\mathcal{M}^1 = (\mathcal{M}_G^1, \mathcal{M}_H^1)$, $\mathcal{M}^2 = (\mathcal{M}_G^2, \mathcal{M}_H^2) \in \mathcal{C}_{\text{w-rad}}$ and suppose $\Xi_{\mathcal{M}^1}^{\min}(\mathcal{M}_H^1) = \mathcal{M}_G^1$. Let $(\cdot, \cdot)^G$ and $\cdot(\cdot, \cdot)^G$ be invariant sesquilinear forms on $\mathcal{M}_G^1 \times \mathcal{M}_G^2$ and $(\cdot, \cdot)^H$ an invariant sesquilinear form on $\mathcal{M}_H^1 \times \mathcal{M}_H^2$. Suppose both pairs $((\cdot, \cdot)^G, (\cdot, \cdot)^H)$ and $(\cdot(\cdot, \cdot)^G, (\cdot, \cdot)^H)$ are compatible with restriction in the sense of Definition 9.9. Then $(\cdot, \cdot)^G = \cdot(\cdot, \cdot)^G$.*

Proof. It suffices to show for $E \in \widehat{K}_M$, $\Phi' \in \text{Hom}_K(E, \mathcal{M}_G^1)$ and $\Phi'' \in \text{Hom}_K(E^*, \mathcal{M}_G^2)$

$$\sum_{i=1}^{\nu} (\Phi'[e_i], \Phi''[e_i^*])^G = \sum_{i=1}^{\nu} \cdot(\Phi'[e_i], \Phi''[e_i^*])^G.$$

Here $\{e_1, \dots, e_{\nu}\}$ is a basis of E and $e_i^* = \overline{e_i^*}$ for $i = 1, \dots, \nu$. Since $\Xi_{\mathcal{M}^1}^{\min}(\mathcal{M}_H^1) = \mathcal{M}_G^1$, $\text{Hom}_K(E, \mathcal{M}_G^1)$ is spanned over \mathbb{C} by elements of the form

$$\Phi_1 \circ \Psi_1 \quad \text{with } V \in \widehat{K}_{\text{qsp}}, \Phi_1 \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(V), \mathcal{M}_G^1) \text{ and } \Psi_1 \in \text{Hom}_K(E, P_G(V)).$$

Hence we have only to show

$$(10.1) \quad \sum_{i=1}^{\nu} (\Phi_1 \circ \Psi_1[e_i], \Phi''[e_i^*])^G = \sum_{i=1}^{\nu} \cdot(\Phi_1 \circ \Psi_1[e_i], \Phi''[e_i^*])^G.$$

Take a basis $\{v_1, \dots, v_{m'}, \dots, v_n\}$ of V so that $\{v_1, \dots, v_{m'}\}$ is a basis of V_{single}^M and $v_{m'+1}, \dots, v_n$ are orthogonal to V_{single}^M . Put $v_j^* = \overline{v_j^*}$ for $j = 1, \dots, n$. Then we have

$$\begin{aligned} \text{L.H.S.} &= \sum_{j=1}^n (\Phi_1[v_j], \Phi'' \circ \Psi_1^*[v_j^*])^G && (\because \text{Corollary 10.5}) \\ &= \sum_{j=1}^{m'} (\tilde{\Gamma}_{\mathcal{M}_1}^V(\Phi_1)[v_j], \tilde{\Gamma}_{\mathcal{M}_2}^{V^*}(\Phi'' \circ \Psi_1^*)[v_j^*])^H. && (\because \Phi_1 \in \text{Hom}^{2 \rightarrow 2}) \end{aligned}$$

But since the right-hand side of (10.1) can be calculated in the same way, (10.1) holds. \square

Identifying $\text{Hom}_K(E, P_G(V))$ with $\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V))$ as in Definition 6.6, we can apply the star operation to the latter.

Proposition 10.7. *In addition to E, V suppose $F \in \widehat{K}_M$. Then it holds that $(\Psi_2 \circ \Psi_1)^* = \Psi_1^* \circ \Psi_2^*$ for any $\Psi_1 \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(F))$ and $\Psi_2 \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(F), P_G(V))$.*

Proof. Put

$$\mathcal{A}(G \times_K E) := \{f : G \xrightarrow{\text{analytic}} E; f(gk) = k^{-1}f(g) \text{ for } k \in K\}.$$

Then there is a sesquilinear form (\cdot, \cdot) on $P_G(E^*) \times \mathcal{A}(G \times_K E)_{K\text{-finite}}$ defined by

$$(D \otimes e^*, f) = r(D)(e^*, f(g))|_{g=1}.$$

One easily checks this is invariant and non-degenerate. Let $I \in \text{End}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E^*))$ be the identity map. Then it follows from Corollary 10.5 that for any $\Phi \in \text{Hom}_K(V, \mathcal{A}(G \times_K$

$$E)) \simeq \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(V, \mathcal{A}(G \times_K E)_{K\text{-finite}}),$$

$$\begin{aligned} \sum_{j=1}^n \left(((\Psi_2 \circ \Psi_1)^* - \Psi_1^* \circ \Psi_2^*)[v_j^*], \Phi[v_j] \right) &= \sum_{i=1}^{\nu} \left(I[e_i^*], (\Phi \circ (\Psi_2 \circ \Psi_1) - \Phi \circ \Psi_2 \circ \Psi_1)[e_i] \right) \\ &= 0. \end{aligned}$$

The non-degeneracy of the form implies $(\Psi_2 \circ \Psi_1)^* = \Psi_1^* \circ \Psi_2^*$. \square

Suppose Y and U are finite-dimensional W -modules. Let us define the star operation for $\text{Hom}_{\mathbb{C}}(Y, P_{\mathbf{H}}(U))$.

Definition 10.8 (the star operation). Let $\{y_1, \dots, y_{\mu}\}$ and $\{u_1, \dots, u_m\}$ be bases of Y and U . Let $\{y_1^*, \dots, y_{\mu}^*\} \subset Y^*$ be the dual basis of $\{y_1, \dots, y_{\mu}\}$ and put $y_i^* = \overline{y_i^*}$ ($i = 1, \dots, \mu$). Define $\{u_1^*, \dots, u_m^*\} \subset U^*$ similarly. For a given $\psi \in \text{Hom}_{\mathbb{C}}(Y, P_{\mathbf{H}}(U))$ we take $f_{ij}(\lambda) \in S(\mathfrak{a}_{\mathbb{C}})$ ($1 \leq i \leq \mu, 1 \leq j \leq m$) so that

$$(10.2) \quad \psi[y_i] = \sum_{j=1}^m f_{ij}(\lambda) \otimes u_j \quad \text{for } i = 1, \dots, \mu.$$

We define $\psi^* \in \text{Hom}_{\mathbb{C}}(U^*, P_{\mathbf{H}}(Y^*))$ by

$$\psi^*[u_j^*] = \sum_{i=1}^{\mu} \overline{f_{ij}(-\bar{\lambda})} \otimes y_i^* \quad \text{for } j = 1, \dots, m.$$

Remark 10.9. (i) It is easy to see the definition is independent of the choice of bases.

(ii) The star operation $\psi \mapsto \psi^*$ is involutive.

(iii) If we consider f_{ij} in the definition as elements of $U(\mathfrak{g}_{\mathbb{C}})$, then $f_{ij}^*(\lambda) = \overline{f_{ij}(-\bar{\lambda})}$.

Identifying $\text{Hom}_{\mathbb{C}}(Y, P_{\mathbf{H}}(U))$ with $\text{Hom}_{\mathfrak{a}_{\mathbb{C}}}(P_{\mathbf{H}}(Y), P_{\mathbf{H}}(U))$ as in Definition 6.6, we can apply the star operation to the latter. The next is immediate from Definition 10.8.

Proposition 10.10. *In addition to Y, U suppose Z is a finite-dimensional W -module. Then it holds that $(\psi_2 \circ \psi_1)^* = \psi_1^* \circ \psi_2^*$ for any $\psi_1 \in \text{Hom}_{\mathfrak{a}_{\mathbb{C}}}(P_{\mathbf{H}}(Y), P_{\mathbf{H}}(Z))$ and $\psi_2 \in \text{Hom}_{\mathfrak{a}_{\mathbb{C}}}(P_{\mathbf{H}}(Z), P_{\mathbf{H}}(U))$.*

The operation $\psi \mapsto \psi^*$ has a similar property to Corollary 10.5.

Proposition 10.11. *Suppose $\psi \in \text{Hom}_W(Y, P_{\mathbf{H}}(U))$. Then $\psi^* \in \text{Hom}_W(U^*, P_{\mathbf{H}}(Y^*))$. Furthermore, suppose \mathcal{X}_1 and \mathcal{X}_2 are \mathbf{H} -modules and (\cdot, \cdot) is an invariant sesquilinear form on $\mathcal{X}_1 \times \mathcal{X}_2$. Suppose bases $\{y_i\}, \{y_i^*\}, \{u_j\}$ and $\{u_j^*\}$ are as in Definition 10.8. Then for $\varphi_1 \in \text{Hom}_W(U, \mathcal{X}_1) \simeq \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(U), \mathcal{X}_1)$ and $\varphi_2 \in \text{Hom}_W(Y^*, \mathcal{X}_2) \simeq \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(Y^*), \mathcal{X}_2)$ it holds that*

$$\sum_{i=1}^{\mu} (\varphi_1 \circ \psi[y_i], \varphi_2[y_i^*]) = \sum_{j=1}^m (\varphi_1[u_j], \varphi_2 \circ \psi^*[u_j^*]).$$

Proof. Express $\psi \in \text{Hom}_W(Y, P_{\mathbf{H}}(U))$ as in (10.2) and define $\psi^\vee \in \text{Hom}_{\mathbb{C}}(U^*, P_{\mathbf{H}}(Y^*))$ by

$$\psi^\vee[u^*] = \frac{1}{|W|} \sum_{i=1}^{\mu} \sum_{k=1}^m \sum_{w \in W} (wu^*, u_k) w^{-1} w_0 \overline{f_{ik}(-w_0 \bar{\lambda})} w_0 \otimes y_i^* \quad \text{for } u^* \in U.$$

Since $w^{-1}u_k = \sum_{j=1}^m \overline{(wu_j^*, u_k)} u_j$ for $w \in W$, we have

$$\begin{aligned} \sum_{i=1}^{\mu} (\varphi_1 \circ \psi[y_i], \varphi_2[y_i^*]) &= \sum_{i=1}^{\mu} \left(\sum_{j=1}^m f_{ij}(\lambda) \varphi_1[u_j], \varphi_2[y_i^*] \right) \\ &= \sum_{i=1}^{\mu} \sum_{j=1}^m (\varphi_1[u_j], w_0 \overline{f_{ij}(-w_0 \bar{\lambda})} w_0 \varphi_2[y_i^*]) \\ &= \frac{1}{|W|} \sum_{i=1}^{\mu} \sum_{k=1}^m \sum_{w \in W} (w \varphi_1[w^{-1}u_k], w_0 \overline{f_{ik}(-w_0 \bar{\lambda})} w_0 \varphi_2[y_i^*]) \\ &= \sum_{j=1}^m \frac{1}{|W|} \sum_{i=1}^{\mu} \sum_{k=1}^m \sum_{w \in W} (\varphi_1[u_j], (wu_j^*, u_k) w^{-1} w_0 \overline{f_{ik}(-w_0 \bar{\lambda})} w_0 \varphi_2[y_i^*]) \\ &= \sum_{j=1}^m (\varphi_1[u_j], \varphi_2 \circ \psi^\vee[u_j^*]). \end{aligned}$$

Also, one easily checks $\psi^\vee \in \text{Hom}_W$. Hence the proposition follows if we can show $\psi^\vee = \psi^*$. To do so, for any fixed $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ suppose $\varphi_1 \in \text{Hom}_W(U, B_{\mathbf{H}}(\lambda_0))$ and $\varphi_2 \in \text{Hom}_W(Y^*, B_{\mathbf{H}}(-\bar{\lambda}_0))$.

On the one hand,

$$\begin{aligned} \sum_{j=1}^m \lambda_0 (\varphi_1[u_j], \varphi_2 \circ \psi^\vee[u_j^*])_{-\bar{\lambda}_0}^{\mathbf{H}} &= \sum_{i=1}^{\mu} \lambda_0 (\varphi_1 \circ \psi[y_i], \varphi_2[y_i^*])_{-\bar{\lambda}_0}^{\mathbf{H}} \\ &= \frac{1}{|W|} \sum_{w \in W} \sum_{i=1}^{\mu} \varphi_1 \circ \psi[y_i](w) \overline{\varphi_2[y_i^*](w)} \\ &= \frac{1}{|W|} \sum_{w \in W} \sum_{i=1}^{\mu} \varphi_1 \circ \psi[w^{-1}y_i](1) \overline{\varphi_2[w^{-1}y_i^*](1)} \\ &= \frac{1}{|W|} \sum_{w \in W} \sum_{i=1}^{\mu} \varphi_1 \circ \psi[w^{-1}y_i](1) \overline{\varphi_2[w^{-1}y_i^*](1)}. \end{aligned}$$

But since $\sum_{i=1}^{\mu} w^{-1}y_i \otimes w^{-1}y_i^* = \sum_{i=1}^{\mu} y_i \otimes y_i^*$, the last expression equals

$$\sum_{i=1}^{\mu} \varphi_1 \circ \psi[y_i](1) \overline{\varphi_2[y_i^*](1)} = \sum_{i=1}^{\mu} \sum_{j=1}^m (f_{ij}(\lambda) \varphi_1[u_j])(1) \overline{\varphi_2[y_i^*](1)}$$

$$\begin{aligned}
&= \sum_{i=1}^{\mu} \sum_{j=1}^m \varphi_1[u_j](w_0 f_{ij}(-w_0 \lambda) w_0) \overline{\varphi_2[y_i^*](1)} \\
&= \sum_{i=1}^{\mu} \sum_{j=1}^m f_{ij}(-\lambda_0) \varphi_1[u_j](1) \overline{\varphi_2[y_i^*](1)}.
\end{aligned}$$

One the other hand, take $g_{ij}(\lambda) \in S(\mathfrak{a}_{\mathbb{C}})$ ($1 \leq i \leq \mu, 1 \leq j \leq m$) so that

$$\psi^{\vee}[u_j^*] = \sum_{i=1}^{\mu} g_{ij}(\lambda) \otimes y_i^* \quad \text{for } j = 1, \dots, m,$$

and apply the same calculation as above to the right entry. Then we have

$$\begin{aligned}
\sum_{j=1}^m \lambda_0(\varphi_1[u_j], \varphi_2 \circ \psi^{\vee}[u_j^*])_{-\bar{\lambda}_0}^{\mathbf{H}} &= \frac{1}{|W|} \sum_{w \in W} \sum_{j=1}^m \varphi_1[u_j](w) \overline{\varphi_2 \circ \psi^{\vee}[u_j^*](w)} \\
&= \sum_{j=1}^m \varphi_1[u_j](1) \overline{\varphi_2 \circ \psi^{\vee}[u_j^*](1)} \\
&= \sum_{j=1}^m \sum_{i=1}^{\nu} \varphi_1[u_j](1) \overline{(g_{ij}(\lambda) \varphi_2[y_i^*])(1)} \\
&= \sum_{j=1}^m \sum_{i=1}^{\nu} \varphi_1[u_j](1) \overline{\varphi_2[y_i^*](w_0 g_{ij}(-w_0 \lambda) w_0)} \\
&= \sum_{j=1}^m \sum_{i=1}^{\nu} \overline{g_{ij}(\bar{\lambda}_0)} \varphi_1[u_j](1) \overline{\varphi_2[y_i^*](1)}.
\end{aligned}$$

Therefore

$$(10.3) \quad \sum_{i=1}^{\mu} \sum_{j=1}^m \left(f_{ij}(-\lambda_0) - \overline{g_{ij}(\bar{\lambda}_0)} \right) \varphi_1[u_j](1) \overline{\varphi_2[y_i^*](1)} = 0.$$

Now for $k = 1, \dots, m$ and $\ell = 1, \dots, \mu$ let us take

$$\varphi_1^{(k)} : U \ni u \mapsto \left(w \mapsto \overline{(u_k^*, w^{-1}u)} \right) \in (\mathbb{C}W)^* \simeq B_{\mathbf{H}}(\lambda_0)$$

as $\varphi_1 \in \text{Hom}_W(U, B_{\mathbf{H}}(\lambda_0))$ and

$$\varphi_2^{(\ell)} : Y^* \ni y^* \mapsto \left(w \mapsto (w^{-1}y^*, y_{\ell}) \right) \in (\mathbb{C}W)^* \simeq B_{\mathbf{H}}(-\bar{\lambda}_0)$$

as $\varphi_2 \in \text{Hom}_W(Y^*, B_{\mathbf{H}}(-\bar{\lambda}_0))$. Then (10.3) reduces to

$$f_{\ell k}(-\lambda_0) - \overline{g_{\ell k}(\bar{\lambda}_0)} = 0$$

since $\varphi_1^{(k)}[u_j](1) = \delta_{kj}$ and $\varphi_2^{(\ell)}[y_i^*](1) = \delta_{\ell i}$. Since λ_0 is arbitrary, $g_{\ell k}(\lambda) = \overline{f_{\ell k}(-\bar{\lambda})}$ for each k and ℓ . This shows $\psi^{\vee} = \psi^*$. \square

The following theorem shows how the two star operations (Definitions 10.2 and 10.8) relate under the functor $\Gamma : \{P_G(V)\} \rightarrow \{P_{\mathbf{H}}(V^M)\}$ (§6).

Theorem 10.12. *Suppose $E, V \in \widehat{K}_M$. For any $\Psi \in \text{Hom}_K(E, P_G(V))$*

$$(10.4) \quad \Gamma_{E^*}^{V^*}(\Psi^*) = \Gamma_V^E(\Psi)^*,$$

$$(10.5) \quad \tilde{\Gamma}_{E^*}^{V^*}(\Psi^*) = \tilde{\Gamma}_V^E(\Psi)^*.$$

Moreover we have

$$(10.6) \quad \text{Hom}_K^{1 \rightarrow 1}(P_G(E), P_G(V))^* = \text{Hom}_K^{2 \rightarrow 2}(P_G(V^*), P_G(E^*)),$$

$$(10.7) \quad \text{Hom}_K^{2 \rightarrow 2}(P_G(E), P_G(V))^* = \text{Hom}_K^{1 \rightarrow 1}(P_G(V^*), P_G(E^*)).$$

The proof of the theorem is very similar to that of Proposition 10.11. In both proofs the key tools are invariant sesquilinear forms for principal series representations. This method is inspired by Kostant's argument in [Ko2, Chapter I, §8] where he shows a theorem essentially equivalent to (10.4) for $V = \mathbb{C}_{\text{triv}}$.

Proof of Theorem 10.12. Let $\{e_1, \dots, e_\mu\}$ and $\{e_{\mu+1}, \dots, e_n\}$ be bases of E^M and $(E^M)^\perp$. Let $\{e_i^*\} \subset E^*$ be as in Proposition 10.3. Take bases $\{v_1, \dots, v_m\} \sqcup \{v_{m+1}, \dots, v_n\} \subset V$ and $\{v_j^*\} \subset V^*$ similarly. Choose $f_{ij} \in U(\mathfrak{n}_\mathbb{C} + \mathfrak{a}_\mathbb{C})$ ($1 \leq i \leq \nu, 1 \leq j \leq n$) so that

$$\Psi[e_i] = \sum_{j=1}^n f_{ij} \otimes v_j \quad \text{for } i = 1, \dots, \nu$$

and $g_{ij} \in U(\mathfrak{n}_\mathbb{C} + \mathfrak{a}_\mathbb{C})$ ($1 \leq i \leq \nu, 1 \leq j \leq n$) so that

$$\Psi^*[v_j^*] = \sum_{i=1}^\nu g_{ij} \otimes e_i^* \quad \text{for } j = 1, \dots, n.$$

Then

$$\begin{aligned} \Gamma_V^E(\Psi)[e_i] &= \sum_{j=1}^m \gamma(f_{ij}) \otimes v_j \quad \text{for } i = 1, \dots, \mu, \\ \Gamma_{E^*}^{V^*}(\Psi^*)[v_j^*] &= \sum_{i=1}^\mu \gamma(g_{ij}) \otimes e_i^* \quad \text{for } j = 1, \dots, m. \end{aligned}$$

Now fix any $\lambda_0 \in \mathfrak{a}_\mathbb{C}^*$ and suppose $\Phi_1 \in \text{Hom}_K(V, B_G(\lambda_0))$, $\Phi_2 \in \text{Hom}_K(E^*, B_G(-\bar{\lambda}_0))$. By a similar calculation to the proofs of Propositions 9.10 and 10.11

$$\begin{aligned} \sum_{i=1}^\nu {}_{\lambda_0}(\Phi_1 \circ \Psi[e_i], \Phi_2[e_i^*])_{-\bar{\lambda}_0}^G &= \sum_{i=1}^\mu \Phi_1 \circ \Psi[e_i](1) \overline{\Phi_2[e_i^*](1)} \\ &= \sum_{i=1}^\mu \sum_{j=1}^n (\ell(f_{ij}) \Phi_1[v_j])(1) \overline{\Phi_2[e_i^*](1)} \\ &= \sum_{i=1}^\mu \sum_{j=1}^n \gamma(f_{ij})(-\lambda_0) \Phi_1[v_j](1) \overline{\Phi_2[e_i^*](1)} \\ &= \sum_{i=1}^\mu \sum_{j=1}^m \gamma(f_{ij})(-\lambda_0) \Phi_1[v_j](1) \overline{\Phi_2[e_i^*](1)}. \end{aligned}$$

Here in the first and last equalities we used respectively

$$\Phi_2[e_i^*](1) = 0 \text{ for } i > \mu, \quad \Phi_1[v_j](1) = 0 \text{ for } j > m$$

(see the proof of Theorem 9.8). But by virtue of Corollary 10.5 this also equals

$$\begin{aligned} \sum_{j=1}^n {}_{\lambda_0}(\Phi_1[v_j], \Phi_2 \circ \Psi^*[v_j^*])_{-\bar{\lambda}_0}^G &= \sum_{j=1}^m \Phi_1[v_j](1) \overline{\Phi_2 \circ \Psi^*[v_j^*](1)} \\ &= \sum_{j=1}^m \sum_{i=1}^{\nu} \Phi_1[v_j](1) \overline{(\ell(g_{ij})\Phi_2[e_i^*])(1)} \\ &= \sum_{j=1}^m \sum_{i=1}^{\mu} \overline{\gamma(g_{ij})(\bar{\lambda}_0)} \Phi_1[v_j](1) \overline{\Phi_2[e_i^*](1)}. \end{aligned}$$

Therefore

$$(10.8) \quad \sum_{i=1}^{\mu} \sum_{j=1}^m \left(\gamma(f_{ij})(-\lambda_0) - \overline{\gamma(g_{ij})(\bar{\lambda}_0)} \right) \Phi_1[v_j](1) \overline{\Phi_2[e_i^*](1)} = 0.$$

Now for $k = 1, \dots, m$ and $\ell = 1, \dots, \mu$ let us take

$$\Phi_1^{(k)} : V \ni v \mapsto \left(k \mapsto \overline{(v_k^*, k^{-1}v)} \right) \in C^\infty(K/M) \simeq B_G(\lambda_0)$$

as $\Phi_1 \in \text{Hom}_K(V, B_G(\lambda_0))$ and

$$\Phi_2^{(\ell)} : E^* \ni e^* \mapsto \left(k \mapsto (k^{-1}e^*, e_\ell) \right) \in C^\infty(K/M) \simeq B_G(-\bar{\lambda}_0)$$

as $\Phi_2 \in \text{Hom}_K(E^*, B_G(-\bar{\lambda}_0))$. Then (10.8) reduces to

$$\gamma(f_{\ell k})(-\lambda_0) - \overline{\gamma(g_{\ell k})(\bar{\lambda}_0)} = 0$$

since $\Phi_1^{(k)}[v_j](1) = \delta_{kj}$ and $\Phi_2^{(\ell)}[e_i^*](1) = \delta_{\ell i}$. Since λ_0 is arbitrary, $\gamma(g_{\ell k})(\lambda) = \overline{\gamma(f_{\ell k})(-\bar{\lambda})}$ for each $k = 1, \dots, m$ and $\ell = 1, \dots, \mu$. This shows (10.4), from which one can easily deduce (10.5)–(10.7). \square

Corollary 10.13. *In Theorem 6.5 the assertions (ii) and (iii) are equivalent via star operations. Hence (iii) is valid.*

11. MODULE STRUCTURE OF $\mathcal{A}(A, \lambda)$

Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and let us study the structure of

$$\mathcal{A}(A, \lambda) = \{ \varphi \in C^\infty(A); \mathcal{T}(\Delta)\varphi = \Delta(\lambda)\varphi \text{ for } \Delta \in S(\mathfrak{a}_{\mathbb{C}})^W \}$$

as an \mathbf{H} -module. Since the center of \mathbf{H} is $S(\mathfrak{a}_{\mathbb{C}})^W$, this is actually an \mathbf{H} -submodule of $C^\infty(A)$. (Recall $h \in \mathbf{H}$ acts on $\varphi \in C^\infty(A)$ by $\mathcal{T}(\theta_{\mathbf{H}}h)\varphi$.) As it turned out to be in §5 and Example 8.25, $\mathcal{A}(A, \lambda)$ is the radial counterpart of

$$\mathcal{A}(G/K, \lambda) = \{ f \in C^\infty(G/K); r(\Delta)f = \gamma(\Delta)(\lambda)f \text{ for } \Delta \in U(\mathfrak{g}_{\mathbb{C}})^K \}.$$

So we review some fundamental facts of $\mathcal{A}(G/K, \lambda)$ first. Throughout the section we fix a non-zero vector v_{triv} of the trivial representation \mathbb{C}_{triv} of K or W . Let $(\cdot, \cdot)_r^G$ be the sesquilinear form on $C^\infty(G/K) \times P_G(\mathbb{C}_{\text{triv}})$ defined by

$$(11.1) \quad (f, D \otimes v_{\text{triv}})_r^G = r(\bar{D})f(1) = \ell(D^*)f(1) \quad \text{for } f \in C^\infty(G/K) \text{ and } D \in U(\mathfrak{g}_{\mathbb{C}})$$

(the conjugation $\bar{\cdot}$ is with respect to the real form \mathfrak{g}). This is clearly $(\mathfrak{g}_{\mathbb{C}}, K)$ -invariant, and non-degenerate when $C^\infty(G/K)$ is replaced with the space $\mathcal{A}(G/K)$ of analytic functions on G/K . The following results are well known:

Proposition 11.1. (i) *All functions in $\mathcal{A}(G/K, \lambda)$ are analytic on G/K .*
(ii) *Put*

$$P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}) = P_G(\mathbb{C}_{\text{triv}}) / \sum_{\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K} U(\mathfrak{g}_{\mathbb{C}})(\Delta - \gamma(\Delta)(\bar{\lambda})) \otimes v_{\text{triv}}.$$

Then this is a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module with a K -invariant cyclic vector. Any K -invariant vector in $P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ is a scalar multiple of $1 \otimes v_{\text{triv}}$. An invariant non-degenerate sesquilinear form on $\mathcal{A}(G/K, \lambda) \times P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ is induced from $(\cdot, \cdot)_r^G$.

(iii) *By (ii), any non-zero $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule of $\mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ contains the spherical function ϕ_λ . Hence there is a unique irreducible submodule $X_G(\lambda) \subset \mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ generated by ϕ_λ .*

Using $\mathcal{A}(G/K, \lambda)$ as a model case, we introduce an invariant sesquilinear form on $C^\infty(A) \times P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})$ defined by

$$(11.2) \quad (\varphi, h \otimes v_{\text{triv}})_{\mathcal{J}}^{\mathbf{H}} = \mathcal{J}(\theta_{\mathbf{H}}(h^*))\varphi(1) \quad \text{for } \varphi \in C^\infty(A) \text{ and } h \in \mathbf{H},$$

and an \mathbf{H} -module

$$P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}) = P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}) / \sum_{\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W} \mathbf{H}(\Delta - \Delta(\bar{\lambda})) \otimes v_{\text{triv}}.$$

Note (11.2) is simply rewritten as

$$(11.3) \quad (\varphi, f \otimes v_{\text{triv}})_{\mathcal{J}}^{\mathbf{H}} = \mathcal{J}(\overline{f(\bar{\cdot})})\varphi(1) \quad \text{for } \varphi \in C^\infty(A) \text{ and } f \in S(\mathfrak{a}_{\mathbb{C}}).$$

Opdam's *non-symmetric hypergeometric functions* are key tools of our investigation:

Theorem 11.2 ([Op1]). *For any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ there exists a unique analytic function $\mathbf{G}(\lambda, x)$ on A satisfying*

$$(11.4) \quad \begin{cases} \mathcal{J}(\xi)\mathbf{G}(\lambda, x) = \lambda(\xi)\mathbf{G}(\lambda, x) & \text{for } \xi \in \mathfrak{a}_{\mathbb{C}}, \\ \mathbf{G}(\lambda, 1) = 1. \end{cases}$$

Under the identification $A \simeq \mathfrak{a}$ by (1.15), there exists an open neighborhood U at $0 \in \mathfrak{a}$ such that $\mathbf{G}(\lambda, x)$ extends to a holomorphic function on $\mathfrak{a}_{\mathbb{C}}^ \times (\mathfrak{a} + iU)$.*

The uniqueness assures that an analytic function satisfying the first condition of (11.4) is a scalar multiple of $\mathbf{G}(\lambda, x)$. In Appendix A we prove a C^∞ function satisfying the first condition of (11.4) is necessarily analytic. Hence we get

Lemma 11.3. *If $\varphi(x) \in C^\infty(A)$ satisfies*

$$\mathcal{T}(\xi)\varphi = \lambda(\xi)\varphi \quad \text{for } \xi \in \mathfrak{a}_{\mathbb{C}},$$

then it is a scalar multiple of $\mathbf{G}(\lambda, x)$. In particular, if moreover $\varphi \neq 0$ then $\varphi(1) \neq 0$.

Observe that $\mathbf{G}(w\lambda, x) \in \mathcal{A}(A, \lambda)$ for any $w \in W$.

Theorem 11.4. (i) *All functions in $\mathcal{A}(A, \lambda)$ are analytic on A .*

(ii) *As a W -module $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ is isomorphic to the regular representation of W . A non-zero W -invariant vector (unique up to a scalar multiple) spans $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ as an \mathbf{H} -module. An invariant non-degenerate sesquilinear form on $\mathcal{A}(A, \lambda) \times P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ is induced from $(\cdot, \cdot)_{\mathcal{T}}^{\mathbf{H}}$.*

(iii) *By (ii), any non-zero \mathbf{H} -submodule of $\mathcal{A}(A, \lambda)$ contains the restriction $\gamma_0(\phi_\lambda)$ of the spherical function (the Heckman-Opdam hypergeometric function with a special parameter). Hence there is a unique irreducible submodule $X_{\mathbf{H}}(\lambda) \subset \mathcal{A}(A, \lambda)$ generated by $\gamma_0(\phi_\lambda)$.*

(iv) *$\mathcal{A}(A, \lambda)$ is spanned by $\mathbf{G}(\lambda, x)$ as an \mathbf{H} -module if and only if*

$$(11.5) \quad \lambda(\alpha^\vee) \neq -\mathbf{m}_1(\alpha) \quad \text{for any } \alpha \in R_1^+,$$

where $\alpha^\vee := \frac{2H_\alpha}{|\alpha|^2}$ is the coroot for α .

(v) *Let \mathbb{C}_{sgn} be the sign representation of W and v_{sgn} its fixed generator. Put*

$$P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}, -\lambda) = P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}) / \sum_{\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W} \mathbf{H}(\Delta - \Delta(-\lambda)) \otimes v_{\text{sgn}}.$$

Then $\mathcal{A}(A, \lambda) \simeq P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}, -\lambda)$ as an \mathbf{H} -module.

Remark 11.5. Essentially the same result as (iv) and the duality between $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ and $P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}, -\lambda)$ are stated in [Ch2].

Proof of Theorem 11.4. First, the identification $S(\mathfrak{a}_{\mathbb{C}}) \xrightarrow{\sim} P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}); f \mapsto f \otimes v_{\text{triv}}$ induces the identification

$$S(\mathfrak{a}_{\mathbb{C}}) / \sum_{\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W} S(\mathfrak{a}_{\mathbb{C}})(\Delta - \Delta(\bar{\lambda})) \xrightarrow{\sim} P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$$

as $S(\mathfrak{a}_{\mathbb{C}})$ -modules. The left-hand side is isomorphic to the space $H_W(\mathfrak{a}_{\mathbb{C}})$ of W -harmonic polynomials on \mathfrak{a}^* as a \mathbb{C} -linear space since

$$S(\mathfrak{a}_{\mathbb{C}}) = H_W(\mathfrak{a}_{\mathbb{C}}) \otimes S(\mathfrak{a}_{\mathbb{C}})^W = H_W(\mathfrak{a}_{\mathbb{C}}) \oplus \sum_{\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W} S(\mathfrak{a}_{\mathbb{C}})(\Delta - \Delta(\bar{\lambda})).$$

But since we also have the decomposition

$$S(\mathfrak{a}_{\mathbb{C}}) = H_W(\mathfrak{a}_{\mathbb{C}}) \oplus S(\mathfrak{a}_{\mathbb{C}})(S(\mathfrak{a}_{\mathbb{C}})\mathfrak{a}_{\mathbb{C}})^W,$$

which is compatible with the decomposition to the homogeneous parts, it holds that for $d = 0, 1, 2, \dots$

$$\begin{aligned} \mathcal{F}^d P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}) &:= \{f \otimes v_{\text{triv}} \in P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}); f \in S(\mathfrak{a}_{\mathbb{C}}) \text{ with } \deg f \leq d\} \\ &= \{f \otimes v_{\text{triv}} \in P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}); f \in H_W(\mathfrak{a}_{\mathbb{C}}) \text{ with } \deg f \leq d\}. \end{aligned}$$

Using these subspaces as a filtered W -module structure of $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$, we get the W -module isomorphisms

$$P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}) \simeq \text{gr}_{\mathcal{F}} P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}) \simeq H_W(\mathfrak{a}_{\mathbb{C}}) \simeq \mathbb{C}W.$$

Likewise $P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}, -\lambda) \simeq \mathbb{C}W \otimes \mathbb{C}_{\text{sgn}} \simeq \mathbb{C}W$ as W -modules.

Secondly, we assert that $\mathbb{C} \prod_{\alpha \in R_1^+} (\alpha^\vee + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}} \subset P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ is a unique W -submodule isomorphic to \mathbb{C}_{sgn} and that

$$P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}) = \mathcal{F}^{|R_1^+|-1} P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}) \oplus \mathbb{C} \prod_{\alpha \in R_1^+} (\alpha^\vee + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}$$

is the direct sum decomposition as a W -module. Indeed, since $\prod_{\alpha \in R_1^+} \alpha^\vee \in H_W(\mathfrak{a}_{\mathbb{C}})$,

$$\prod_{\alpha \in R_1^+} (\alpha^\vee + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}} \equiv \prod_{\alpha \in R_1^+} \alpha^\vee \otimes v_{\text{triv}} \not\equiv 0 \pmod{\mathcal{F}^{|R_1^+|-1} P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})}.$$

Moreover for any $\beta \in \Pi$ we have $\prod_{\alpha \in R_1^+ \setminus \{\beta\}} (\alpha^\vee + \mathbf{m}_1(\alpha)) \in S(\mathfrak{a}_{\mathbb{C}})^{s_\beta} = S(\mathfrak{a}_{\mathbb{C}}^{s_\beta}) \cdot \mathbb{C}[(\beta^\vee)^2]$ where $\mathfrak{a}_{\mathbb{C}}^{s_\beta} = \{\xi \in \mathfrak{a}_{\mathbb{C}}; \beta(\xi) = 0\}$. By using (4.1) one can check $S(\mathfrak{a}_{\mathbb{C}})^{s_\beta}$ commutes with s_β also in \mathbf{H} . Hence using (4.1) again we calculate

$$\begin{aligned} s_\beta \prod_{\alpha \in R_1^+} (\alpha^\vee + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}} &= \left(\prod_{\alpha \in R_1^+ \setminus \{\beta\}} (\alpha^\vee + \mathbf{m}_1(\alpha)) \right) s_\beta (\beta^\vee + \mathbf{m}_1(\beta)) \otimes v_{\text{triv}} \\ &= \left(\prod_{\alpha \in R_1^+ \setminus \{\beta\}} (\alpha^\vee + \mathbf{m}_1(\alpha)) \right) (-\beta^\vee s_\beta - \mathbf{m}_1(\beta)(2 - s_\beta)) \otimes v_{\text{triv}} \\ &= - \prod_{\alpha \in R_1^+} (\alpha^\vee + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}. \end{aligned}$$

This shows $\mathbb{C} \prod_{\alpha \in R_1^+} (\alpha^\vee + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}} \simeq \mathbb{C}_{\text{sgn}}$. The other assertions are obvious.

Thirdly, let us prove any non-zero \mathbf{H} -submodule of $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ contains $\prod_{\alpha \in R_1^+} (\alpha^\vee + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}$. For this purpose take any non-zero $f \in H_W(\mathfrak{a}_{\mathbb{C}})$. Let f' denote the highest homogeneous part of f . By the theory of W -harmonic polynomials there exists a homogeneous $g \in S(\mathfrak{a}_{\mathbb{C}})$ such that $\partial(g)\partial(f') \prod_{\alpha \in R_1^+} \alpha = 1$. For such g we easily see

$$\sum_{w \in W} (\text{sgn } w) g(w \cdot) f(w \cdot) = \sum_{w \in W} (\text{sgn } w) g(w \cdot) f'(w \cdot) = c \prod_{\alpha \in R_1^+} \alpha^\vee \quad \text{with } c \neq 0.$$

Hence in $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ it holds that

$$\sum_{w \in W} (\text{sgn } w) g(w \cdot) w^{-1} (f \otimes v_{\text{triv}}) \equiv c \prod_{\alpha \in R_1^+} \alpha^\vee \otimes v_{\text{triv}} \not\equiv 0 \pmod{\mathcal{F}^{|R_1^+|-1} P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})}.$$

This shows $\mathbf{H}(f \otimes v_{\text{triv}}) \not\subset \mathcal{F}^{|R_1^+|-1} P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$, which, combined with the previous argument, implies $\prod_{\alpha \in R_1^+} (\alpha^\vee + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}} \in \mathbf{H}(f \otimes v_{\text{triv}})$.

Fourthly, it is clear from (11.3) that $(\cdot, \cdot)_{\mathcal{F}}^{\mathbf{H}}$ induces an invariant sesquilinear form on $\mathcal{A}(A, \lambda) \times P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ (we use the same symbol $(\cdot, \cdot)_{\mathcal{F}}^{\mathbf{H}}$ for this form). We assert this is non-degenerate. Indeed, if $\varphi \in \mathcal{A}(A, \lambda)$ is non-zero then $\mathcal{T}(S(\mathfrak{a}_{\mathbb{C}}))\varphi = \mathcal{T}(H_W(\mathfrak{a}_{\mathbb{C}}))\varphi$

has finite dimension and is annihilated by $\mathcal{T}(\Delta) - \Delta(\lambda)$ for any $\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W$. Hence there exist some $f \in S(\mathfrak{a}_{\mathbb{C}})$ and $w \in W$ such that $\mathcal{T}(f)\varphi \neq 0$ and

$$\mathcal{T}(\xi)\mathcal{T}(f)\varphi = (w\lambda)(\xi)\mathcal{T}(f)\varphi \quad \text{for any } \xi \in \mathfrak{a}_{\mathbb{C}}.$$

By Lemma 11.3 $\mathcal{T}(f)\varphi$ is a non-zero multiple of $\mathbf{G}(w\lambda, x)$. Thus it holds that

$$(\varphi, \overline{f(\cdot)} \otimes v_{\text{triv}})_{\mathcal{T}}^{\mathbf{H}} = \mathcal{T}(f)\varphi(1) \neq 0.$$

Conversely, if $D \in P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ is non-zero then there exists some $h \in \mathbf{H}$ such that

$$hD = \prod_{\alpha \in R_1^+} (\alpha^{\vee} + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}.$$

Since for any $w \in W$

$$(11.6) \quad \left(\mathbf{G}(w\lambda, x), \prod_{\alpha \in R_1^+} (\alpha^{\vee} + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}} \right)_{\mathcal{T}}^{\mathbf{H}} = \prod_{\alpha \in R_1^+} ((w\lambda)(\alpha^{\vee}) + \mathbf{m}_1(\alpha))$$

and this is non-zero for a suitable choice of $w \in W$, we get for such w

$$(h^* \mathbf{G}(w\lambda, x), D)_{\mathcal{T}}^{\mathbf{H}} = \left(\mathbf{G}(w\lambda, x), \prod_{\alpha \in R_1^+} (\alpha^{\vee} + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}} \right)_{\mathcal{T}}^{\mathbf{H}} \neq 0.$$

This completes the proof of (ii) and hence (iii).

Fifthly, since $\mathbf{H} = \theta_{\mathbf{H}}(\mathbb{C}W S(\mathfrak{a}_{\mathbb{C}})) = \mathbb{C}W \theta_{\mathbf{H}}(S(\mathfrak{a}_{\mathbb{C}}))$, we have

$$\mathcal{T}(\theta_{\mathbf{H}}\mathbf{H}) \mathbf{G}(\lambda, x) = \mathbb{C}W \mathcal{T}(S(\mathfrak{a}_{\mathbb{C}})) \mathbf{G}(\lambda, x) = \mathbb{C}W \mathbf{G}(\lambda, x).$$

Hence, $\mathbf{G}(\lambda, x)$ spans $\mathcal{A}(A, \lambda)$ if and only if the orthogonal complement of $\mathbb{C}W \mathbf{G}(\lambda, x)$ with respect to the sesquilinear form does not contain $\prod_{\alpha \in R_1^+} (\alpha^{\vee} + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}$, if and only if $t \mathbf{G}(\lambda, x)$ and $\prod_{\alpha \in R_1^+} (\alpha^{\vee} + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}$ are not orthogonal for some $t \in W$, if and only if $\mathbf{G}(\lambda, x)$ and $\prod_{\alpha \in R_1^+} (\alpha^{\vee} + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}$ are not orthogonal, if and only if (11.6) with $w = 1$ is non-zero, if and only if (11.5) is satisfied. Thus (iv) is proved. If we choose $w \in W$ so that (11.6) is non-zero then $\mathcal{A}(A, \lambda) = \mathcal{A}(A, w\lambda)$ is spanned by the analytic function $\mathbf{G}(w\lambda, x)$. This implies (i).

Finally it follows from the non-degeneracy of the sesquilinear form that there exists a non-zero $\phi_{\text{sgn}} \in \mathcal{A}(A, \lambda)$ such that $\mathbb{C}\phi_{\text{sgn}} \simeq \mathbb{C}_{\text{sgn}}$ as a W -module. Such ϕ_{sgn} is unique up to a non-zero scalar and is not orthogonal to $\prod_{\alpha \in R_1^+} (\alpha^{\vee} + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}$. Now any $\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W$ acts on $\mathcal{A}(A, \lambda)$ by

$$\mathcal{T}(\theta_{\mathbf{H}}\Delta) = \mathcal{T}(w_0 \Delta(-w_0 \cdot) w_0) = \mathcal{T}(w_0 \Delta(-\cdot) w_0) = \mathcal{T}(\Delta(-\cdot)) = \Delta(-\lambda).$$

Hence by the obvious universal property of $P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}, -\lambda)$ there exists a unique \mathbf{H} -homomorphism $P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}, -\lambda) \rightarrow \mathcal{A}(A, \lambda)$ such that $1 \otimes v_{\text{sgn}} \mapsto \phi_{\text{sgn}}$. This is surjective since the orthogonal complement of $\mathcal{T}(\theta_{\mathbf{H}}\mathbf{H}) \phi_{\text{sgn}}$ does not contain $\prod_{\alpha \in R_1^+} (\alpha^{\vee} + \mathbf{m}_1(\alpha)) \otimes v_{\text{triv}}$. The injectivity is clear from the dimension argument. Thus we get (v). \square

As we saw in Example 8.25, $(\mathcal{A}(G/K, \lambda)_{K\text{-finite}}, \mathcal{A}(A, \lambda))$ is a radial pair. So each \mathbf{H} -submodule of $\mathcal{A}(A, \lambda)$ is lifted to a $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodule of $\mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ by the correspondence Ξ_0^{\min} introduced in Definition 8.21.

Theorem 11.6. $\Xi_0^{\min}(X_{\mathbf{H}}(\lambda)) = X_G(\lambda)$. In particular $(X_G(\lambda), X_{\mathbf{H}}(\lambda))$ is a radial pair.

Proof. We apply Proposition 8.11 to the case where $\mathcal{M} = (\mathcal{A}(G/K, \lambda)_{K\text{-finite}}, \mathcal{A}(A, \lambda))$ and $\phi_W = \gamma_0(\phi_\lambda)$. Thus $\phi_K = \phi_\lambda$ and there exists a morphism

$$\mathcal{I} = (\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : (P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})) \rightarrow (\mathcal{A}(G/K, \lambda)_{K\text{-finite}}, \mathcal{A}(A, \lambda))$$

in \mathcal{C}_{rad} such that $\mathcal{I}_G(1 \otimes v_{\text{triv}}) = \phi_\lambda$ and $\mathcal{I}_{\mathbf{H}}(1 \otimes v_{\text{triv}}) = \gamma_0(\phi_\lambda)$. Here clearly $\text{Im } \mathcal{I} = (X_G(\lambda), X_{\mathbf{H}}(\lambda))$ and $\Xi^{\min}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})) = P_G(\mathbb{C}_{\text{triv}})$. Hence by (8.15) we have

$$\Xi_0^{\min}(X_{\mathbf{H}}(\lambda)) = \Xi_0^{\min}(\mathcal{I}_{\mathbf{H}}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))) = \mathcal{I}_G(\Xi^{\min}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))) = \mathcal{I}_G(P_G(\mathbb{C}_{\text{triv}})) = X_G(\lambda). \quad \square$$

12. POISSON TRANSFORMS

Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. The Poisson transform \mathcal{P}_G^λ is the G -homomorphism of $B_G(\lambda)$ into $\mathcal{A}(G/K, \lambda)$ defined by

$$\mathcal{P}_G^\lambda : B_G(\lambda) \ni F(g) \longmapsto \int_K F(xk) dk \in \mathcal{A}(G/K, \lambda).$$

To construct an analogous \mathbf{H} -homomorphism of $B_{\mathbf{H}}(\lambda)$ into $\mathcal{A}(A, \lambda)$ we recall another description of \mathcal{P}_G^λ according to [Hel5, Ch. II, §3, No. 4]. For $g \in G$ let $A(g)$ be the unique element of \mathfrak{a} such that $g = n \exp A(g) k$ for some $n \in N$ and $k \in K$. Then the function

$$G \times G \ni (g, x) \longmapsto e^{(\bar{\lambda} + \rho)(A(g^{-1}x))} \in \mathbb{C}$$

belongs to $B_G(-\bar{\lambda})$ as a function in g and to $\mathcal{A}(G/K, \bar{\lambda})$ as a function in x . If ${}_{\lambda}(\cdot, \cdot)_{-\bar{\lambda}}^G$ is the invariant sesquilinear form of Definition 9.1 then for $F(g) \in B_G(\lambda)$

$$\begin{aligned} \mathcal{P}_G^\lambda F(x) &= \int_K F(xk) dk = \int_K F(xk) \overline{e^{(\bar{\lambda} + \rho)(A(k^{-1}x))}} dk \\ &= {}_{\lambda}(F(xg), e^{(\bar{\lambda} + \rho)(A(g^{-1}x))})_{-\bar{\lambda}}^G = {}_{\lambda}(F(g), e^{(\bar{\lambda} + \rho)(A((x^{-1}g)^{-1}x))})_{-\bar{\lambda}}^G \\ &= {}_{\lambda}(F(g), e^{(\bar{\lambda} + \rho)(A(g^{-1}x))})_{-\bar{\lambda}}^G = \int_K F(k) \overline{e^{(\bar{\lambda} + \rho)(A(k^{-1}x))}} dk \\ &= \int_K F(k) e^{(\lambda + \rho)(A(k^{-1}x))} dk. \end{aligned}$$

Now the function

$$\mathbf{H} \times A \ni (h, x) \longmapsto \mathcal{T}(\theta_{\mathbf{H}} h) \mathbf{G}(\bar{\lambda}, x) \in \mathbb{C}$$

belongs to $B_{\mathbf{H}}(-\bar{\lambda})$ as a function in h and to $\mathcal{A}(A, \bar{\lambda})$ as a function in x . If ${}_{\lambda}(\cdot, \cdot)_{-\bar{\lambda}}^{\mathbf{H}}$ is the invariant sesquilinear form of Definition 9.3 then for $F(h) \in B_{\mathbf{H}}(\lambda)$

$$\begin{aligned} {}_{\lambda}(F(h), \mathcal{T}(\theta_{\mathbf{H}} h) \mathbf{G}(\bar{\lambda}, x))_{-\bar{\lambda}}^{\mathbf{H}} &= \frac{1}{|W|} \sum_{w \in W} F(w) \overline{\mathcal{T}(\theta_{\mathbf{H}} w) \mathbf{G}(\bar{\lambda}, x)} \\ (12.1) \qquad &= \frac{1}{|W|} \sum_{w \in W} F(w) \mathbf{G}(\lambda, w^{-1}x). \end{aligned}$$

Here we used the relation $\overline{\mathbf{G}(\bar{\lambda}, x)} = \mathbf{G}(\lambda, x)$, which is obvious from (11.4). Since the final expression of (12.1) belongs to $\mathcal{A}(A, \lambda)$, we define the Poisson transform for $B_{\mathbf{H}}(\lambda)$ by

$$(12.2) \quad \mathcal{P}_{\mathbf{H}}^{\lambda} : B_{\mathbf{H}}(\lambda) \ni F(h) \mapsto \frac{1}{|W|} \sum_{w \in W} F(w) \mathbf{G}(\lambda, w^{-1}x) \in \mathcal{A}(A, \lambda).$$

This is an \mathbf{H} -homomorphism since for any $a \in \mathbf{H}$

$$\begin{aligned} {}_{\lambda}(F({}^t a h), \mathcal{T}(\theta_{\mathbf{H}} h) \mathbf{G}(\bar{\lambda}, x))_{-\bar{\lambda}}^{\mathbf{H}} &= {}_{\lambda}(F(h), \mathcal{T}(\theta_{\mathbf{H}}({}^t(a^*)h)) \mathbf{G}(\bar{\lambda}, x))_{-\bar{\lambda}}^{\mathbf{H}} \\ &= {}_{\lambda}(F(h), \mathcal{T}(\overline{\theta_{\mathbf{H}} a}) \mathcal{T}(\theta_{\mathbf{H}} h) \mathbf{G}(\bar{\lambda}, x))_{-\bar{\lambda}}^{\mathbf{H}} \\ &= \mathcal{T}(\theta_{\mathbf{H}} a) {}_{\lambda}(F(h), \mathcal{T}(\theta_{\mathbf{H}} h) \mathbf{G}(\bar{\lambda}, x))_{-\bar{\lambda}}^{\mathbf{H}}. \end{aligned}$$

Proposition 12.1. *The Poisson transform $\mathcal{P}_{\mathbf{H}}^{\lambda}$ is bijective if and only if (11.5) is satisfied. This condition is rewritten as*

$$\begin{cases} \lambda(\alpha^{\vee}) + \dim \mathfrak{g}_{\alpha} \neq 0 & \text{for } \alpha \in \Sigma^+ \setminus (2\Sigma^+ \cup \frac{1}{2}\Sigma^+), \\ \lambda(\alpha^{\vee}) + \dim \mathfrak{g}_{\alpha} + 2 \dim \mathfrak{g}_{2\alpha} \neq 0 & \text{for } \alpha \in \Sigma^+ \cap \frac{1}{2}\Sigma^+. \end{cases}$$

Remark 12.2. (i) The surjectivity of $\mathcal{P}_{\mathbf{H}}^{\lambda}$ is equivalent to its injectivity since $\dim B_{\mathbf{H}}(\lambda) = \dim \mathcal{A}(A, \lambda) = |W|$.

(ii) This condition is much weaker than the following well-known condition for the bijectivity of $\mathcal{P}_G^{\lambda} : B_G(\lambda)_{K\text{-finite}} \rightarrow \mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ (cf. [Hel2]):

$$\begin{cases} \lambda(\alpha^{\vee}) + \dim \mathfrak{g}_{\alpha} \notin \{0, -2, -4, \dots\} & \text{for } \alpha \in \Sigma^+ \setminus (2\Sigma^+ \cup \frac{1}{2}\Sigma^+), \\ \lambda(\alpha^{\vee}) + \dim \mathfrak{g}_{\alpha} \notin \{-2, -6, -10, \dots\} & \text{for } \alpha \in \Sigma^+ \cap \frac{1}{2}\Sigma^+, \\ \lambda(\alpha^{\vee}) + \dim \mathfrak{g}_{\alpha} + 2 \dim \mathfrak{g}_{2\alpha} \notin \{0, -4, -8, \dots\} & \text{for } \alpha \in \Sigma^+ \cap \frac{1}{2}\Sigma^+. \end{cases}$$

Proof of Proposition 12.1. By (12.2) we have

$$\begin{aligned} \text{Im } \mathcal{P}_{\mathbf{H}}^{\lambda} &= \sum_{w \in W} \mathbb{C} \mathbf{G}(\lambda, w^{-1}x) = \sum_{w \in W} (\mathcal{T}(S(\mathfrak{a}_{\mathbb{C}})) \mathbf{G})(\lambda, w^{-1}x) \\ &= \mathcal{T}(WS(\mathfrak{a}_{\mathbb{C}})) \mathbf{G}(\lambda, x) = \mathcal{T}(\mathbf{H}) \mathbf{G}(\lambda, x) = \mathcal{T}(\theta_{\mathbf{H}} \mathbf{H}) \mathbf{G}(\lambda, x). \end{aligned}$$

Hence the proposition follows from Theorem 11.4 (iv). \square

The following is the main result of this section:

Theorem 12.3. *The pair of $\mathcal{P}_G^{\lambda} : B_G(\lambda)_{K\text{-finite}} \rightarrow \mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ and $\mathcal{P}_{\mathbf{H}}^{\lambda} : B_{\mathbf{H}}(\lambda) \rightarrow \mathcal{A}(A, \lambda)$ is a morphism of radial pairs:*

$$\mathcal{P}^{\lambda} = (\mathcal{P}_G^{\lambda}, \mathcal{P}_{\mathbf{H}}^{\lambda}) : (B_G(\lambda)_{K\text{-finite}}, B_{\mathbf{H}}(\lambda)) \longrightarrow (\mathcal{A}(G/K, \lambda)_{K\text{-finite}}, \mathcal{A}(A, \lambda)).$$

That is, if $V \in \hat{K}_M$ then

(i) *for any $\Phi \in \text{Hom}_K(V, B_G(\lambda))$ the two maps*

$$\begin{aligned} V_{\text{single}}^M &\hookrightarrow V \xrightarrow{\Phi} B_G(\lambda) \xrightarrow{\mathcal{P}_G^{\lambda}} \mathcal{A}(G/K, \lambda) \xrightarrow{\gamma_0} C^{\infty}(A), \\ V_{\text{single}}^M &\hookrightarrow V \xrightarrow{\Phi} B_G(\lambda) \xrightarrow{\gamma_{B(\lambda)}} B_{\mathbf{H}}(\lambda) \xrightarrow{\mathcal{P}_{\mathbf{H}}^{\lambda}} \mathcal{A}(A, \lambda) \hookrightarrow C^{\infty}(A) \end{aligned}$$

coincide and

(ii) for any $\Phi \in \text{Hom}_K(V, B_G(\lambda))$ such that

$$\Phi[v](1) = 0 \quad \forall v \in V_{\text{double}}^M$$

it holds that

$$\mathcal{P}_G^\lambda(\Phi[v])(x) = 0 \quad \forall v \in V_{\text{double}}^M \text{ and } \forall x \in A.$$

These properties can be more explicitly stated as follows:

(i') For any $\Phi \in \text{Hom}_K(V, C^\infty(K/M))$ it holds that

$$\int_K \Phi[v](k) e^{(\lambda+\rho)(A(k^{-1}x))} dk = \frac{1}{|W|} \sum_{w \in W} \Phi[v](\bar{w}) \mathbf{G}(\lambda, w^{-1}x) \quad \forall v \in V_{\text{single}}^M \text{ and } \forall x \in A$$

($\bar{w} \in N_K(\mathfrak{a})$ is a lift of w) and

(ii') for any $\Phi \in \text{Hom}_K(V, C^\infty(K/M))$ such that

$$\Phi[v](1) = 0 \quad \forall v \in V_{\text{double}}^M$$

it holds that

$$\int_K \Phi[v](k) e^{(\lambda+\rho)(A(k^{-1}x))} dk = 0 \quad \forall v \in V_{\text{double}}^M \text{ and } \forall x \in A.$$

The proof requires some preparation.

Proposition 12.4. For $(C^\infty(G/K)_{K\text{-finite}}, C^\infty(A))$ and $(P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$, the pair of invariant sesquilinear forms $(\cdot, \cdot)_r^G$ and $(\cdot, \cdot)_{\mathcal{F}}^{\mathbf{H}}$ is compatible with restriction in the sense of Definition 9.9.

Proof. Suppose $V \in \widehat{K}_{\text{qsp}}$ and take bases $\{v_1, \dots, v_{m'}, \dots, v_n\} \subset V$ and $\{v_1^*, \dots, v_n^*\} \subset V^*$ as in Definition 9.9. Let $I_G \in \text{End}_{\mathfrak{g}_{\mathbf{C}}, K}(P_G(\mathbb{C}_{\text{triv}}))$ be the identity and take $v_{\text{triv}}^* \in \mathbb{C}_{\text{triv}}^*$ so that $(v_{\text{triv}}^*, v_{\text{triv}}) = 1$. For $(\Phi, \Psi) \in \text{Hom}_K(V, C^\infty(G/K)) \times \text{Hom}_K(V^*, P_G(\mathbb{C}_{\text{triv}}))$

$$\begin{aligned} \sum_{i=1}^n (\Phi[v_i], \Psi[v_i^*])_r^G &= \sum_{i=1}^n (\Phi[v_i], I_G \circ \Psi[v_i^*])_r^G \\ &= (\Phi \circ \Psi^*[v_{\text{triv}}^*], I_G[v_{\text{triv}}])_r^G \quad (\because \text{Corollary 10.5}) \\ &= (\Phi \circ \Psi^*[v_{\text{triv}}^*], 1 \otimes v_{\text{triv}})_r^G \\ &= \Phi \circ \Psi^*[v_{\text{triv}}^*](1) \quad (\because (11.1)) \\ &= \tilde{\Gamma}_0^{\mathbb{C}_{\text{triv}}^*}(\Phi \circ \Psi^*)[v_{\text{triv}}^*](1). \end{aligned}$$

Here if $(\Phi, \Psi) \in \text{Hom}_K^{2 \rightarrow 2} \times \text{Hom}_K$ then

$$(12.3) \quad \tilde{\Gamma}_0^{\mathbb{C}_{\text{triv}}^*}(\Phi \circ \Psi^*) = \tilde{\Gamma}_0(\Phi) \circ \tilde{\Gamma}_V^{\mathbb{C}_{\text{triv}}^*}(\Psi^*)$$

by Theorem 7.3 (i). Formula (12.3) also holds for $(\Phi, \Psi) \in \text{Hom}_K \times \text{Hom}_K^{2 \rightarrow 2}$ since in this case $\Psi^* \in \text{Hom}_K^{1 \rightarrow 1}$ by (10.7) and Theorem 7.3 (ii) can be applied. Hence in either case, letting $I_{\mathbf{H}} \in \text{End}_{\mathfrak{g}_{\mathbf{C}}, K}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$ be the identity we have

$$\sum_{i=1}^n (\Phi[v_i], \Psi[v_i^*])_r^G = \tilde{\Gamma}_0(\Phi) \circ \tilde{\Gamma}_V^{\mathbb{C}_{\text{triv}}^*}(\Psi^*)[v_{\text{triv}}^*](1)$$

$$\begin{aligned}
&= (\tilde{\Gamma}_0(\Phi) \circ \tilde{\Gamma}_V^{\mathbb{C}^* \text{triv}}(\Psi^*)[v_{\text{triv}}^*], 1 \otimes v_{\text{triv}})_{\mathcal{J}}^{\mathbf{H}} \quad (\because (11.2)) \\
&= (\tilde{\Gamma}_0(\Phi) \circ \tilde{\Gamma}_V^{\mathbb{C}^* \text{triv}}(\Psi^*)[v_{\text{triv}}^*], I_{\mathbf{H}}[v_{\text{triv}}])_{\mathcal{J}}^{\mathbf{H}} \\
&= \sum_{i=1}^{m'} (\tilde{\Gamma}_0^V(\Phi)[v_i], I_{\mathbf{H}} \circ (\tilde{\Gamma}_V^{\mathbb{C}^* \text{triv}}(\Psi^*))^*[v_i^*])_{\mathcal{J}}^{\mathbf{H}} \quad (\because \text{Proposition 10.11}) \\
&= \sum_{i=1}^{m'} (\tilde{\Gamma}_0^V(\Phi)[v_i], \tilde{\Gamma}_{\mathbb{C}^* \text{triv}}^{V^*}(\Psi)[v_i^*])_{\mathcal{J}}^{\mathbf{H}} \quad (\because (10.5))
\end{aligned}$$

and the proposition. \square

Let $\mathbf{1}_G^{-\bar{\lambda}} \in B_G(-\bar{\lambda})$ and $\mathbf{1}_{\mathbf{H}}^{-\bar{\lambda}} \in B_{\mathbf{H}}(-\bar{\lambda})$ denote functions which constantly take 1 on K and W respectively. Then $\mathbf{1}_G^{-\bar{\lambda}}$ and $\mathbf{1}_{\mathbf{H}}^{-\bar{\lambda}}$ are K - and W -invariant respectively. Since $\gamma_{B(-\bar{\lambda})}(\mathbf{1}_G^{-\bar{\lambda}}) = \mathbf{1}_{\mathbf{H}}^{-\bar{\lambda}}$ it follows from Proposition 8.11 that there exists a morphism $\mathcal{I}^{-\bar{\lambda}} = (\mathcal{I}_G^{-\bar{\lambda}}, \mathcal{I}_{\mathbf{H}}^{-\bar{\lambda}}) : (P_G(\mathbb{C}^* \text{triv}), P_{\mathbf{H}}(\mathbb{C}^* \text{triv})) \rightarrow (B_G(-\bar{\lambda}), B_{\mathbf{H}}(-\bar{\lambda}))$ in \mathcal{C}_{rad} such that $\mathcal{I}_G^{-\bar{\lambda}}(1 \otimes v_{\text{triv}}) = \mathbf{1}_G^{-\bar{\lambda}}$ and $\mathcal{I}_{\mathbf{H}}^{-\bar{\lambda}}(1 \otimes v_{\text{triv}}) = \mathbf{1}_{\mathbf{H}}^{-\bar{\lambda}}$.

Lemma 12.5.

$$\begin{aligned}
(\mathcal{P}_G^\lambda F, D \otimes v_{\text{triv}})_r^G &= {}_\lambda(F, \mathcal{I}_G^{-\bar{\lambda}}(D \otimes v_{\text{triv}}))_{-\bar{\lambda}}^G \quad \text{for } F \in B_G(\lambda) \text{ and } D \in U(\mathfrak{g}_{\mathbb{C}}), \\
(\mathcal{P}_{\mathbf{H}}^\lambda F, h \otimes v_{\text{triv}})_{\mathcal{J}}^{\mathbf{H}} &= {}_\lambda(F, \mathcal{I}_{\mathbf{H}}^{-\bar{\lambda}}(h \otimes v_{\text{triv}}))_{-\bar{\lambda}}^{\mathbf{H}} \quad \text{for } F \in B_{\mathbf{H}}(\lambda) \text{ and } h \in \mathbf{H}.
\end{aligned}$$

Proof. The first formula holds since

$$\begin{aligned}
(\mathcal{P}_G^\lambda F, D \otimes v_{\text{triv}})_r^G &= (D^* \mathcal{P}_G^\lambda F, 1 \otimes v_{\text{triv}})_r^G = (\mathcal{P}_G^\lambda(D^* F), 1 \otimes v_{\text{triv}})_r^G \\
&= \mathcal{P}_G^\lambda(D^* F)(1) = {}_\lambda((D^* F)(g), e^{(\bar{\lambda}+\rho)(A(g^{-1}))})_{-\bar{\lambda}}^G \\
&= {}_\lambda(D^* F, \mathbf{1}_G^{-\bar{\lambda}})_{-\bar{\lambda}}^G = {}_\lambda(D^* F, \mathcal{I}_G^{-\bar{\lambda}}(1 \otimes v_{\text{triv}}))_{-\bar{\lambda}}^G \\
&= {}_\lambda(F, D \mathcal{I}_G^{-\bar{\lambda}}(1 \otimes v_{\text{triv}}))_{-\bar{\lambda}}^G = {}_\lambda(F, \mathcal{I}_G^{-\bar{\lambda}}(D \otimes v_{\text{triv}}))_{-\bar{\lambda}}^G.
\end{aligned}$$

One can deduce the second formula by an analogous calculation. \square

Lemma 12.6. Suppose $\mathcal{M}^1 = (\mathcal{M}_G^1, \mathcal{M}_{\mathbf{H}}^1)$, $\mathcal{M}^2 = (\mathcal{M}_G^2, \mathcal{M}_{\mathbf{H}}^2) \in \mathcal{C}_{\text{Ch}}$ and a pair of invariant sesquilinear forms $(\cdot, \cdot)_{\mathcal{M}}^G$ on $\mathcal{M}_G^1 \times \mathcal{M}_G^2$ and $(\cdot, \cdot)_{\mathcal{M}}^{\mathbf{H}}$ on $\mathcal{M}_{\mathbf{H}}^1 \times \mathcal{M}_{\mathbf{H}}^2$ is compatible with restriction. Suppose $\mathcal{N}^1 = (\mathcal{N}_G^1, \mathcal{N}_{\mathbf{H}}^1)$, $\mathcal{N}^2 = (\mathcal{N}_G^2, \mathcal{N}_{\mathbf{H}}^2) \in \mathcal{C}_{\text{Ch}}$ and a pair of invariant sesquilinear forms $(\cdot, \cdot)_{\mathcal{N}}^G$ on $\mathcal{N}_G^1 \times \mathcal{N}_G^2$ and $(\cdot, \cdot)_{\mathcal{N}}^{\mathbf{H}}$ on $\mathcal{N}_{\mathbf{H}}^1 \times \mathcal{N}_{\mathbf{H}}^2$ is compatible with restriction. Suppose $\mathcal{I}^2 = (\mathcal{I}_G^2, \mathcal{I}_{\mathbf{H}}^2) : \mathcal{N}^2 \rightarrow \mathcal{M}^2$ is a morphism in \mathcal{C}_{Ch} . Suppose \mathcal{I}_G^2 and $\mathcal{I}_{\mathbf{H}}^2$ have thier adjoint \mathcal{I}_G^1 and $\mathcal{I}_{\mathbf{H}}^1$ satisfying

$$\begin{aligned}
(\mathcal{I}_G^1(y_1), y_2)_{\mathcal{N}}^G &= (y_1, \mathcal{I}_G^2(y_2))_{\mathcal{M}}^G \quad \text{for } y_1 \in \mathcal{M}_G^1 \text{ and } y_2 \in \mathcal{N}_G^2, \\
(\mathcal{I}_{\mathbf{H}}^1(x_1), x_2)_{\mathcal{N}}^{\mathbf{H}} &= (x_1, \mathcal{I}_{\mathbf{H}}^2(x_2))_{\mathcal{M}}^{\mathbf{H}} \quad \text{for } x_1 \in \mathcal{M}_{\mathbf{H}}^1 \text{ and } x_2 \in \mathcal{N}_{\mathbf{H}}^2.
\end{aligned}$$

Finally suppose $(\mathcal{N}_G^2)^\perp = \{0\}$ and $(\mathcal{N}_{\mathbf{H}}^2)^\perp = \{0\}$, that is

$$(12.4) \quad \{y_1 \in \mathcal{N}_G^1; (y_1, y_2)_{\mathcal{N}}^G = 0 \text{ for any } y_2 \in \mathcal{N}_G^2\} = \{0\},$$

$$(12.5) \quad \{x_1 \in \mathcal{N}_{\mathbf{H}}^1; (x_1, x_2)_{\mathcal{N}}^{\mathbf{H}} = 0 \text{ for any } x_2 \in \mathcal{N}_{\mathbf{H}}^2\} = \{0\}.$$

Then \mathcal{I}_G^1 and \mathcal{I}_H^1 constitute a morphism $\mathcal{I}^1 := (\mathcal{I}_G^1, \mathcal{I}_H^1) : \mathcal{M}^1 \rightarrow \mathcal{N}^1$ in \mathcal{C}_{Ch} .

Proof. We have to check \mathcal{I}^1 satisfies Conditions (Ch-1) and (Ch-2) in Definition 8.1. To do so, let $V \in \widehat{K}_M$ be arbitrary and take bases $\{v_1, \dots, v_{m'}, \dots, v_n\} \subset V$ and $\{v_i^*\} \subset V^*$ as in Definition 9.9. We first confirm (Ch-1), namely $\tilde{\Gamma}_{\mathcal{N}^1}^V(\mathcal{I}_G^1 \circ \Phi_1) = \mathcal{I}_H^1 \circ \tilde{\Gamma}_{\mathcal{M}^1}^V(\Phi_1)$ for any $\Phi_1 \in \text{Hom}_K(V, \mathcal{M}_G^1)$. Because of (12.5) it is enough to show

$$\sum_{i=1}^{m'} (\tilde{\Gamma}_{\mathcal{N}^1}^V(\mathcal{I}_G^1 \circ \Phi_1)[v_i], \varphi_2[v_i^*])_{\mathcal{N}}^{\mathbf{H}} = \sum_{i=1}^{m'} (\mathcal{I}_H^1 \circ \tilde{\Gamma}_{\mathcal{M}^1}^V(\Phi_1)[v_i], \varphi_2[v_i^*])_{\mathcal{N}}^{\mathbf{H}}$$

for any $\varphi_2 \in \text{Hom}_W((V^*)_{\text{single}}^M, \mathcal{N}_H^2)$. But if we take $\Phi_2 \in \text{Hom}_K^{2 \rightarrow 2}(V^*, \mathcal{N}_G^2)$ so that $\tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi_2) = \varphi_2$ then

$$\begin{aligned} & \sum_{i=1}^{m'} (\tilde{\Gamma}_{\mathcal{N}^1}^V(\mathcal{I}_G^1 \circ \Phi_1)[v_i], \varphi_2[v_i^*])_{\mathcal{N}}^{\mathbf{H}} \\ &= \sum_{i=1}^n (\mathcal{I}_G^1 \circ \Phi_1[v_i], \Phi_2[v_i^*])_{\mathcal{N}}^G \\ &= \sum_{i=1}^n (\Phi_1[v_i], \mathcal{I}_G^2 \circ \Phi_2[v_i^*])_{\mathcal{M}}^G \\ &= \sum_{i=1}^{m'} (\tilde{\Gamma}_{\mathcal{M}^1}^V(\Phi_1)[v_i], \tilde{\Gamma}_{\mathcal{M}^2}^{V^*}(\mathcal{I}_G^2 \circ \Phi_2)[v_i])_{\mathcal{M}}^{\mathbf{H}} \quad (\because \mathcal{I}_G^2 \circ \Phi_2 \in \text{Hom}_K^{2 \rightarrow 2}) \\ &= \sum_{i=1}^{m'} (\tilde{\Gamma}_{\mathcal{M}^1}^V(\Phi_1)[v_i], \mathcal{I}_H^2 \circ \tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi_2)[v_i])_{\mathcal{M}}^{\mathbf{H}} \quad (\because \text{(Ch-1) for } \mathcal{I}^2) \\ &= \sum_{i=1}^{m'} (\mathcal{I}_H^1 \circ \tilde{\Gamma}_{\mathcal{M}^1}^V(\Phi_1)[v_i], \varphi_2[v_i^*])_{\mathcal{N}}^{\mathbf{H}}. \end{aligned}$$

Thus \mathcal{I}^1 satisfies Condition (Ch-1).

Secondly we assert

$$(12.6) \quad \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{N}_G^1) = \left\{ \Phi_1 \in \text{Hom}_K(V, \mathcal{N}_G^1); \begin{array}{l} \sum_{i=1}^n (\Phi_1[v_i], \Phi_2[v_i^*])_{\mathcal{N}}^G = 0 \\ \text{for all } \Phi_2 \in \text{Hom}_K(V^*, \mathcal{N}_G^2) \\ \text{such that } \tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi_2) = 0 \end{array} \right\}.$$

Indeed ‘ \subset ’ is immediate from the compatibility of the sesquilinear forms with restriction. In order to show the inverse inclusion, take any Φ_1 in the right-hand side of (12.6). Then there exists a unique $\Phi'_1 \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{N}_G^1)$ such that $\tilde{\Gamma}_{\mathcal{N}^1}^V(\Phi_1) = \tilde{\Gamma}_{\mathcal{N}^1}^V(\Phi'_1)$. Now suppose $\Phi_2 \in \text{Hom}_K(V^*, \mathcal{N}_G^2)$ is arbitrary and take $\Phi'_2 \in \text{Hom}_K^{2 \rightarrow 2}(V^*, \mathcal{N}_G^2)$ so that $\tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi_2) = \tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi'_2)$. Since $\tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi_2 - \Phi'_2) = 0$ and $\Phi_1 - \Phi'_1$ belongs to the right-hand

side of (12.6), we have

$$\begin{aligned}
& \sum_{i=1}^n ((\Phi_1 - \Phi'_1)[v_i], \Phi_2[v_i^*])_{\mathcal{N}}^G \\
&= \sum_{i=1}^n ((\Phi_1 - \Phi'_1)[v_i], (\Phi_2 - \Phi'_2)[v_i^*])_{\mathcal{N}}^G + \sum_{i=1}^n ((\Phi_1 - \Phi'_1)[v_i], \Phi'_2[v_i^*])_{\mathcal{N}}^G \\
&= 0 + \sum_{i=1}^{m'} (\tilde{\Gamma}_{\mathcal{N}^1}^V(\Phi_1 - \Phi'_1)[v_i], \tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi'_2)[v_i^*])_{\mathcal{N}}^{\mathbf{H}} \\
&= 0.
\end{aligned}$$

Hence $\Phi_1 - \Phi'_1 = 0$ by (12.4), proving (12.6).

Let us prove (Ch-2) for \mathcal{I}^1 . Suppose $\Phi_1 \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G^1)$. Then for any $\Phi_2 \in \text{Hom}_K(V^*, \mathcal{N}_G^2)$ with $\tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi_2) = 0$ it holds that

$$\begin{aligned}
\sum_{i=1}^n (\mathcal{I}_G^1 \circ \Phi_1[v_i], \Phi_2[v_i^*])_{\mathcal{N}}^G &= \sum_{i=1}^n (\Phi_1[v_i], \mathcal{I}_G^2 \circ \Phi_2[v_i^*])_{\mathcal{M}}^G \\
&= \sum_{i=1}^{m'} (\tilde{\Gamma}_{\mathcal{M}^1}^V(\Phi_1)[v_i], \tilde{\Gamma}_{\mathcal{M}^2}^{V^*}(\mathcal{I}_G^2 \circ \Phi_2)[v_i^*])_{\mathcal{M}}^{\mathbf{H}} \\
&= \sum_{i=1}^{m'} (\tilde{\Gamma}_{\mathcal{M}^1}^V(\Phi_1)[v_i], \mathcal{I}_{\mathbf{H}}^2 \circ \tilde{\Gamma}_{\mathcal{N}^2}^{V^*}(\Phi_2)[v_i^*])_{\mathcal{M}}^{\mathbf{H}} \quad (\because \text{(Ch-1) for } \mathcal{I}^2) \\
&= 0.
\end{aligned}$$

Hence $\mathcal{I}_G^1 \circ \Phi_1 \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{N}_G^1)$ by (12.6). \square

Proof of Theorem 12.3. We have only to apply Lemma 12.6 to the case where

$$\begin{aligned}
\mathcal{M}^1 &= (B_G(\lambda)_{K\text{-finite}}, B_{\mathbf{H}}(\lambda)), & \mathcal{M}^2 &= (B_G(-\bar{\lambda})_{K\text{-finite}}, B_{\mathbf{H}}(-\bar{\lambda})), \\
\mathcal{N}^1 &= (\mathcal{A}(G/K, \lambda)_{K\text{-finite}}, \mathcal{A}(A, \lambda)), & \mathcal{N}^2 &= (P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})), \\
\mathcal{I}^1 &= (\mathcal{P}_G^\lambda, \mathcal{P}_{\mathbf{H}}^\lambda), & \mathcal{I}^2 &= \mathcal{I}^{-\bar{\lambda}}.
\end{aligned}$$

\square

The following property of the Poisson transforms will be used in the next section:

Proposition 12.7.

$$\begin{aligned}
\text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(B_G(\lambda)_{K\text{-finite}}, \mathcal{A}(G/K, \lambda)_{K\text{-finite}}) &= \mathbb{C} \mathcal{P}_G^\lambda, \\
\text{Hom}_{\mathbf{H}}(B_{\mathbf{H}}(\lambda), \mathcal{A}(A, \lambda)) &= \mathbb{C} \mathcal{P}_{\mathbf{H}}^\lambda.
\end{aligned}$$

Proof. We only prove the first equality since the second one can be proved in the same way. Suppose $\mathcal{I}_G \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(B_G(\lambda)_{K\text{-finite}}, \mathcal{A}(G/K, \lambda)_{K\text{-finite}})$ is given. Since a K -invariant function in $\mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ is a scalar multiple of $\phi_\lambda = \mathcal{P}_G^\lambda \mathbf{1}_G^\lambda$, there exists a constant c such that $\mathcal{I}_G \mathbf{1}_G^\lambda = c \mathcal{P}_G^\lambda \mathbf{1}_G^\lambda$. Hence $\text{Ker}(\mathcal{I}_G - c \mathcal{P}_G^\lambda) \supset \mathbb{C} \mathbf{1}_G^\lambda$. Since the multiplicity of the trivial K -type in $B_G(\lambda)_{K\text{-finite}}$ is 1, $\text{Im}(\mathcal{I}_G - c \mathcal{P}_G^\lambda)$ does not contain ϕ_λ . It then follows from Proposition 11.1 (iii) that $\text{Im}(\mathcal{I}_G - c \mathcal{P}_G^\lambda) = \{0\}$. This proves $\mathcal{I}_G = c \mathcal{P}_G^\lambda$. \square

We conclude this section by presenting a new series of radial pairs. Put

$$D_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}) := \sum_{\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K} U(\mathfrak{g}_{\mathbb{C}})(\Delta - \gamma(\Delta)(\bar{\lambda})) \otimes v_{\text{triv}} \subset P_G(\mathbb{C}_{\text{triv}}),$$

$$D_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}) := \sum_{\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W} \mathbf{H}(\Delta - \Delta(\bar{\lambda})) \otimes v_{\text{triv}} \subset P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})$$

and recall

$$P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}) = P_G(\mathbb{C}_{\text{triv}})/D_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}), \quad P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}) = P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})/D_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}).$$

Proposition 12.8. *Let*

$$\Xi^{\natural} : \{\mathbf{H}\text{-submodules of } P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})\} \rightarrow \{(\mathfrak{g}_{\mathbb{C}}, K)\text{-submodules of } P_G(\mathbb{C}_{\text{triv}})\}$$

be the correspondence defined in Proposition 8.26 (iv). Then

$$\Xi^{\natural}(D_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})) = \Xi^{\min}(D_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})) = D_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}).$$

Hence by Proposition 8.26 (iii), $(P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}))$ is a radial pair with a radial restriction satisfying (rest-3).

Proof. For any $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^K$, $(\mathbb{C}_{\text{triv}} \ni cv_{\text{triv}} \mapsto c(\gamma(\Delta) - \gamma(\Delta)(\bar{\lambda})) \otimes v_{\text{triv}} \in P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})) \in \text{Hom}_W(\mathbb{C}_{\text{triv}}, P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$ is lifted to $(\mathbb{C}_{\text{triv}} \ni cv_{\text{triv}} \mapsto c(\Delta - \gamma(\Delta)(\bar{\lambda})) \otimes v_{\text{triv}} \in P_G(\mathbb{C}_{\text{triv}})) \in \text{Hom}_K^{2 \rightarrow 2}(\mathbb{C}_{\text{triv}}, P_G(\mathbb{C}_{\text{triv}}))$. Hence

$$D_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}) \subset \Xi^{\min}(D_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})) \subset \Xi^{\natural}(D_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})).$$

Suppose $w \in W$ and put

$$U(w\lambda) := \{D \in P_G(\mathbb{C}_{\text{triv}}); (\mathcal{P}_G^{w\lambda}F, D)_r^G = 0 \text{ for any } F \in B_G(w\lambda)\}.$$

Let us prove

(12.7)

$$U(w\lambda) = \sum \{\mathcal{V} \subset P_G(\mathbb{C}_{\text{triv}}); \text{ a } K\text{-stable subspace with } \gamma(D)(w\bar{\lambda}) = 0 \text{ for } \forall D \in \mathcal{V}\}$$

where we identify $P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})$ with $S(\mathfrak{a}_{\mathbb{C}})$ as in Example 6.1. Since both sides of (12.7) are $(\mathfrak{g}_{\mathbb{C}}, K)$ -submodules of $P_G(\mathbb{C}_{\text{triv}})$, it suffices to show for any $V \in \widehat{K}_M$

$$\text{Hom}_K(V, U(w\lambda)) = \{\Phi \in \text{Hom}_K(V, P_G(\mathbb{C}_{\text{triv}})); \gamma(\Phi[v])(w\bar{\lambda}) = 0 \text{ for } v \in V\}.$$

Let $\Phi \in \text{Hom}_K(V, P_G(\mathbb{C}_{\text{triv}}))$ be arbitrary. If we choose $D \in U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})$ for $v \in V$ so that $\Phi[v] = D \otimes v_{\text{triv}}$ then

$$(\mathcal{I}_G^{-w\bar{\lambda}} \circ \Phi[v])(1) = \mathcal{I}_G^{-w\bar{\lambda}}[D \otimes v_{\text{triv}}](1) = (D \mathbf{1}_G^{-w\bar{\lambda}})(1) = \gamma(D)(w\bar{\lambda}) = \gamma(\Phi[v])(w\bar{\lambda}).$$

Hence we have

$$\gamma(\Phi[v])(w\bar{\lambda}) = 0 \quad \text{for any } v \in V$$

$$\iff (\mathcal{I}_G^{-w\bar{\lambda}} \circ \Phi[k^{-1}v])(1) = 0 \quad \text{for any } v \in V \text{ and } k \in K$$

$$\iff (\mathcal{I}_G^{-w\bar{\lambda}} \circ \Phi[v])(k) = 0 \quad \text{for any } v \in V \text{ and } k \in K$$

$$\iff {}_{w\lambda}(F, \mathcal{I}_G^{-w\bar{\lambda}} \circ \Phi[v])_{-w\bar{\lambda}}^G = 0 \quad \text{for any } v \in V \text{ and } F \in B_G(w\lambda)$$

$$\begin{aligned} &\iff (\mathcal{P}_G^{w\lambda} F, \Phi[v])_r^G = 0 \quad \text{for any } v \in V \text{ and } F \in B_G(w\lambda) \quad (\because \text{Lemma 12.5}) \\ &\iff \Phi \in \text{Hom}_K(V, U(w\lambda)). \end{aligned}$$

Thus we get (12.7). Hence in particular $\Xi^\natural(D_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})) \subset U(w\lambda)$.

Now choose $w \in W$ so that

$$\mathcal{P}_G^{w\lambda}(B_G(w\lambda)_{K\text{-finite}}) = \mathcal{A}(G/K, w\lambda)_{K\text{-finite}} = \mathcal{A}(G/K, \lambda)_{K\text{-finite}}$$

(cf. Remark 12.2 (ii)). Then $D_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}) = U(w\lambda)$ by Proposition 11.1 (ii). Hence

$$\Xi^\natural(D_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})) \subset U(w\lambda) = D_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}). \quad \square$$

13. INTERTWINING OPERATORS

For an arbitrary $w \in W$ let $\bar{w} \in N_K(\mathfrak{a})$ be its lift. Let $d\bar{n}$ be a Haar measure of $\bar{w}^{-1}N\bar{w} \cap \theta N$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, [KuS] shows the intertwining operator $\mathcal{A}_G(w, \lambda) : B_G(\lambda) \rightarrow B_G(w\lambda)$ formally given by

$$\mathcal{A}_G(w, \lambda)F(g) = \int_{\bar{w}^{-1}N\bar{w} \cap \theta N} F(g\bar{w}\bar{n}) d\bar{n}$$

converges and makes sense when $-\text{Re } \lambda$ is sufficiently dominant. (By [Sch] the integral is convergent when λ satisfies $\text{Re } \lambda(\alpha^\vee) < 0$ for all $\alpha \in \Sigma^+ \cap -w^{-1}\Sigma^+$.) This operator clearly does not depend on the choice of \bar{w} . In [KnS1] Knapp and Stein prove that $\mathcal{A}_G(w, \lambda)$, as an operator acting on $C^\infty(K/M) \simeq B_G(\lambda)$ with the holomorphic parameter λ , extends meromorphically in λ to the whole $\mathfrak{a}_{\mathbb{C}}^*$. Now let us assume for each $\alpha \in \Pi$ the Haar measure $d\bar{n}$ of $\bar{s}_\alpha^{-1}N\bar{s}_\alpha \cap \theta N$ is normalized so that

$$\int_{\bar{s}_\alpha^{-1}N\bar{s}_\alpha \cap \theta N} e^{2\rho(A(\bar{n}))} d\bar{n} = 1.$$

It then follows from a result of [Sch] and its correction by [KnS2, §2] that we can normalize other Haar measures so that

$$(13.1) \quad \mathcal{A}_G(w, \lambda) = \mathcal{A}_G(w_1, w_2\lambda) \mathcal{A}_G(w_2, \lambda)$$

whenever $w = w_1w_2$ ($w_1, w_2 \in W$) is a minimal decomposition (namely, the length of w is the sum of those of w_1 and w_2) and all intertwining operators in the formula make sense.

Definition 13.1. For $\alpha \in R_1^+$ put

$$\begin{aligned} \mathbf{e}_\alpha(\lambda) &= \left\{ \Gamma\left(\frac{1}{2}\left(\frac{1}{2} \dim \mathfrak{g}_{\alpha/2} + 1 + \lambda(\alpha^\vee)\right)\right) \Gamma\left(\frac{1}{2}(\mathbf{m}_1(\alpha) + \lambda(\alpha^\vee))\right) \right\}^{-1}, \\ \mathbf{c}_\alpha(\lambda) &= 2^{-\lambda(\alpha^\vee)} \Gamma(\lambda(\alpha^\vee)) \mathbf{e}_\alpha(\lambda). \end{aligned}$$

For $\alpha \in \Pi$ define

$$\tilde{\mathcal{A}}_G(s_\alpha, \lambda) = \frac{\mathbf{c}_\alpha(\rho)}{\mathbf{c}_\alpha(-\lambda)} \mathcal{A}_G(s_\alpha, \lambda).$$

Proposition 13.2. (i) For any $\alpha \in \Pi$ the intertwining operator $\tilde{\mathcal{A}}_G(s_\alpha, \lambda) : B_G(\lambda) \rightarrow B_G(s_\alpha \lambda)$ makes sense if and only if $\mathbf{e}_\alpha(-\lambda) \neq 0$. For such λ it holds that

$$(13.2) \quad \tilde{\mathcal{A}}_G(s_\alpha, \lambda) \mathbf{1}_G^\lambda = \mathbf{1}_G^{s_\alpha \lambda}$$

where $\mathbf{1}_G^\lambda \in B_G(\lambda)$ and $\mathbf{1}_G^{s_\alpha \lambda} \in B_G(s_\alpha \lambda)$ are functions taking the constant value 1 on K as in the last section.

(ii) Suppose $w \in W$ and let $w = s_{\alpha_1} \cdots s_{\alpha_k}$ be a reduced expression of w . Then the intertwining operator

$$\tilde{\mathcal{A}}_G(w, \lambda) = \tilde{\mathcal{A}}_G(s_{\alpha_1}, s_{\alpha_2} \cdots s_{\alpha_k} \lambda) \tilde{\mathcal{A}}_G(s_{\alpha_2}, s_{\alpha_3} \cdots s_{\alpha_k} \lambda) \cdots \tilde{\mathcal{A}}_G(s_{\alpha_k}, \lambda)$$

is defined independently of the expression.

(iii) For any $w_1, w_2 \in W$ it holds that

$$\tilde{\mathcal{A}}_G(w_1 w_2, \lambda) = \tilde{\mathcal{A}}_G(w_1, w_2 \lambda) \tilde{\mathcal{A}}_G(w_2, \lambda).$$

Proof. In the proof of (i) we may assume G has real rank 1 without loss of generality (cf. [KnS1]). If $\operatorname{Re} \lambda(\alpha^\vee) < 0$ then we have for $k \in K$

$$(13.3) \quad \begin{aligned} \mathcal{A}_G(s_\alpha, \lambda) \mathbf{1}_G^\lambda(k) &= \int_{\bar{s}_\alpha^{-1} N \bar{s}_\alpha \cap \theta N} \mathbf{1}_G^\lambda(k \bar{s}_\alpha \bar{n}) d\bar{n} = \int_{\bar{s}_\alpha^{-1} N \bar{s}_\alpha \cap \theta N} \mathbf{1}_G^\lambda(\bar{n}) d\bar{n} \\ &= \int_{\bar{s}_\alpha^{-1} N \bar{s}_\alpha \cap \theta N} e^{(\lambda - \rho)(-A(\bar{n}^{-1}))} d\bar{n} = \frac{\mathbf{c}_\alpha(-\lambda)}{\mathbf{c}_\alpha(\rho)}. \end{aligned}$$

Here the last equality follows from [Hel4, Ch. IV, Theorem 6.4]. By analytic continuation, (13.3) is valid whenever $\mathcal{A}_G(s_\alpha, \lambda)$ is defined and (13.2) is valid whenever $\tilde{\mathcal{A}}_G(s_\alpha, \lambda)$ is defined. Recall in general the Poisson transform $\mathcal{P}_G^\lambda : B_G(\lambda)_{K\text{-finite}} \rightarrow \mathcal{A}(G/K, \lambda)_{K\text{-finite}}$ is bijective if and only if $\mathbf{e}_\alpha(\lambda) \neq 0$ (cf. Remark 12.2).

Suppose $\mathbf{e}_\alpha(-\lambda_0) \neq 0$. Since $\mathbf{e}_\alpha(-\bar{\lambda}_0) = \mathbf{e}_\alpha(-\lambda_0) \neq 0$ we have $B_G(-\bar{\lambda}_0)_{K\text{-finite}} \simeq \mathcal{A}(G/K, -\bar{\lambda}_0)_{K\text{-finite}}$ and it follows from Proposition 9.2 and Proposition 11.1 (ii) that the $(\mathfrak{g}_\mathbb{C}, K)$ -module $B_G(\lambda_0)_{K\text{-finite}}$ is equivalent to $P_G(\mathbb{C}_{\text{triv}}, -\lambda_0)$ and is generated by $\mathbf{1}_G^{\lambda_0}$. If λ_0 is a singular point for which $\mathcal{A}_G(s_\alpha, \lambda_0)$ does not make sense then it follows from [KnS1, Theorem 3] that the regularized intertwining operator $(\lambda(\alpha^\vee) - \lambda_0(\alpha^\vee)) \mathcal{A}_G(s_\alpha, \lambda)$ is well defined around $\lambda = \lambda_0$ and non-zero. In this case since

$$\begin{aligned} &(\lambda(\alpha^\vee) - \lambda_0(\alpha^\vee)) \mathcal{A}_G(s_\alpha, \lambda) \Big|_{\lambda=\lambda_0} (B_G(\lambda_0)_{K\text{-finite}}) \\ &= (\lambda(\alpha^\vee) - \lambda_0(\alpha^\vee)) \mathcal{A}_G(s_\alpha, \lambda) \Big|_{\lambda=\lambda_0} (U(\mathfrak{g}_\mathbb{C}) \mathbf{1}_G^{\lambda_0}) \\ &= U(\mathfrak{g}_\mathbb{C}) \left((\lambda(\alpha^\vee) - \lambda_0(\alpha^\vee)) \frac{\mathbf{c}_\alpha(-\lambda)}{\mathbf{c}_\alpha(\rho)} \Big|_{\lambda=\lambda_0} \right) \mathbf{1}_G^{s_\alpha \lambda_0} \neq \{0\} \end{aligned}$$

by (13.3), we can conclude $\lambda = \lambda_0$ is a pole of $\mathbf{c}_\alpha(-\lambda)$ and $\tilde{\mathcal{A}}_G(s_\alpha, \lambda_0)$ is well defined. On the other hand, if $\mathcal{A}_G(s_\alpha, \lambda_0)$ makes sense then $\tilde{\mathcal{A}}_G(s_\alpha, \lambda_0)$ also makes sense since $\mathbf{c}_\alpha(-\lambda)^{-1}$ is regular at $\lambda = \lambda_0$.

Suppose now $\mathbf{e}_\alpha(-\lambda_0) = 0$. Then $\mathcal{P}_G^{\lambda_0} : B_G(\lambda_0)_{K\text{-finite}} \rightarrow \mathcal{A}(G/K, \lambda_0)_{K\text{-finite}}$ is bijective since $\mathbf{e}_\alpha(\lambda_0) \neq 0$ while $\mathcal{P}_G^{s_\alpha \lambda_0} : B_G(s_\alpha \lambda_0)_{K\text{-finite}} \rightarrow \mathcal{A}(G/K, s_\alpha \lambda_0)_{K\text{-finite}}$ is not bijective since $\mathbf{e}_\alpha(s_\alpha \lambda_0) = \mathbf{e}_\alpha(-\lambda_0) = 0$. The latter means $\mathcal{P}_G^{s_\alpha \lambda_0} \Big|_{B_G(s_\alpha \lambda_0)_{K\text{-finite}}}$ is not

surjective. Now assume $\tilde{\mathcal{A}}_G(s_\alpha, \lambda_0)$ makes sense. Then $\mathcal{P}_G^{s_\alpha \lambda_0} \circ \tilde{\mathcal{A}}_G(s_\alpha, \lambda_0) \mathbf{1}_G^{\lambda_0} = \phi_{\lambda_0}$ by (13.2) and it follows from Proposition 12.7 that

$$\mathcal{P}_G^{\lambda_0} \big|_{B_G(\lambda_0)_{K\text{-finite}}} = \mathcal{P}_G^{s_\alpha \lambda_0} \circ \tilde{\mathcal{A}}_G(s_\alpha, \lambda_0) \big|_{B_G(\lambda_0)_{K\text{-finite}}}.$$

This is a contradiction since the left-hand side is bijective while the right-hand side is not. Hence $\tilde{\mathcal{A}}_G(s_\alpha, \lambda_0)$ cannot make sense. Thus we get (i).

Let us return to the general case where G may have higher rank. For a generic λ the right-hand side of the formula of (ii) equals a scalar multiple of $\mathcal{A}_G(w, \lambda)$ by (13.1). This scalar does not depend on the decomposition since $\tilde{\mathcal{A}}_G(w, \lambda) \mathbf{1}_G^\lambda = \mathbf{1}_G^{w\lambda}$ by (i).

To prove (iii) suppose $w_1, w_2 \in W$. Since it is well known that $B_G(\lambda)$ is irreducible when $\lambda \in i\mathfrak{a}^*$,

$$\tilde{\mathcal{A}}_G(w_2, \lambda)^{-1} \tilde{\mathcal{A}}_G(w_1, w_2 \lambda)^{-1} \tilde{\mathcal{A}}_G(w_1 w_2, \lambda) \in \text{End}_G(B_G(\lambda))$$

is a well-defined scalar operator. This sends $\mathbf{1}_G^\lambda$ to itself and hence must be the identity. Thus (iii) is valid for $\lambda \in i\mathfrak{a}^*$, from which the general case follows. \square

Let us develop an analogous story for \mathbf{H} . For $\alpha \in \Pi$ put

$$\tau_\alpha = \alpha^\vee s_\alpha + \mathbf{m}_1(\alpha) \in \mathbf{H}.$$

Then by (4.1) we have

$$\tau_\alpha \xi = s_\alpha(\xi) \tau_\alpha \quad \text{for any } \xi \in \mathfrak{a}_\mathbb{C}^*.$$

Suppose $w \in W$ and let $w = s_{\alpha_1} \cdots s_{\alpha_k}$ be a reduced expression. Then one can prove that

$$\tau_w = \tau_{\alpha_1} \cdots \tau_{\alpha_k} \in \mathbf{H}$$

is defined independently of the expression (see [Op2, Theorem 4.2]). Hence we can define the intertwining operator $\mathcal{A}_\mathbf{H}(w, \lambda) : B_\mathbf{H}(\lambda) \rightarrow B_\mathbf{H}(w\lambda)$ by

$$\mathcal{A}_\mathbf{H}(w, \lambda) F(h) = F(h^t \tau_{w^{-1}}).$$

If $w = w_1 w_2$ is a minimal decomposition then it clearly holds that

$$\mathcal{A}_\mathbf{H}(w, \lambda) = \mathcal{A}_\mathbf{H}(w_1, w_2 \lambda) \mathcal{A}_\mathbf{H}(w_2, \lambda).$$

Definition 13.3. For $\alpha \in \Pi$ define

$$\tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, \lambda) = \frac{1}{\mathbf{m}_1(\alpha) - \lambda(\alpha^\vee)} \mathcal{A}_\mathbf{H}(s_\alpha, \lambda).$$

Proposition 13.4. (i) For any $\alpha \in \Pi$ the intertwining operator $\tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, \lambda) : B_\mathbf{H}(\lambda) \rightarrow B_\mathbf{H}(s_\alpha \lambda)$ makes sense if and only if $\mathbf{m}_1(\alpha) \neq \lambda(\alpha^\vee)$. For such λ it holds that

$$(13.4) \quad \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, \lambda) \mathbf{1}_\mathbf{H}^\lambda = \mathbf{1}_\mathbf{H}^{s_\alpha \lambda}$$

where $\mathbf{1}_\mathbf{H}^\lambda \in B_\mathbf{H}(\lambda)$ and $\mathbf{1}_\mathbf{H}^{s_\alpha \lambda} \in B_\mathbf{H}(s_\alpha \lambda)$ are functions taking the constant value 1 on W .

(ii) Suppose $w \in W$ and let $w = s_{\alpha_1} \cdots s_{\alpha_k}$ be a reduced expression. Then the intertwining operator

$$\tilde{\mathcal{A}}_\mathbf{H}(w, \lambda) = \tilde{\mathcal{A}}_\mathbf{H}(s_{\alpha_1}, s_{\alpha_2} \cdots s_{\alpha_k} \lambda) \tilde{\mathcal{A}}_\mathbf{H}(s_{\alpha_2}, s_{\alpha_3} \cdots s_{\alpha_k} \lambda) \cdots \tilde{\mathcal{A}}_\mathbf{H}(s_{\alpha_k}, \lambda)$$

is defined independently of the expression.

(iii) For any $w_1, w_2 \in W$ it holds that

$$\tilde{\mathcal{A}}_{\mathbf{H}}(w_1 w_2, \lambda) = \tilde{\mathcal{A}}_{\mathbf{H}}(w_1, w_2 \lambda) \tilde{\mathcal{A}}_{\mathbf{H}}(w_2, \lambda).$$

Proof. Suppose $\alpha \in \Pi$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. For $F \in B_{\mathbf{H}}(\lambda)$ one easily calculates

$$(13.5) \quad \mathcal{A}_{\mathbf{H}}(s_{\alpha}, \lambda) F(w) = \mathbf{m}_1(\alpha) F(w) - \lambda(\alpha^{\vee}) F(ws_{\alpha}) \quad \text{for } w \in W.$$

Hence we readily have $\mathcal{A}_{\mathbf{H}}(s_{\alpha}, \lambda) \neq 0$ and $\tilde{\mathcal{A}}_{\mathbf{H}}(s_{\alpha}, \lambda) \mathbf{1}_{\mathbf{H}}^{\lambda} = (\mathbf{m}_1(\alpha) - \lambda(\alpha^{\vee})) \mathbf{1}_{\mathbf{H}}^{s_{\alpha} \lambda}$. These facts prove (i).

We can prove (ii) in the same way as in the proof of Proposition 13.2 (ii).

It is well known that $B_{\mathbf{H}}(\lambda)$ is irreducible if and only if

$$(13.6) \quad \lambda(\alpha^{\vee}) \neq \mathbf{m}_1(\alpha) \quad \text{for any } \alpha \in R_1.$$

Indeed (13.6) is equivalent to the condition that both $\mathcal{P}_{\mathbf{H}}^{\lambda}$ and $\mathcal{P}_{\mathbf{H}}^{-\bar{\lambda}}$ are bijective by Proposition 12.1. The former bijectivity implies any non-zero submodule of $B_{\mathbf{H}}(\lambda)$ contains $\mathbf{1}_{\mathbf{H}}^{\lambda}$ by Theorem 11.4 (iii) and the latter implies $B_{\mathbf{H}}(\lambda) = \mathbf{H} \mathbf{1}_{\mathbf{H}}^{\lambda}$ by Proposition 9.5 and Theorem 11.4 (ii). Thus (13.6) implies the irreducibility of $B_{\mathbf{H}}(\lambda)$. Conversely, if $B_{\mathbf{H}}(\lambda)$ is irreducible then both $\mathcal{P}_{\mathbf{H}}^{\lambda}$ and $\mathcal{P}_{\mathbf{H}}^{-\bar{\lambda}}$ are clearly bijective and (13.6) holds. In particular $B_{\mathbf{H}}(\lambda)$ is irreducible when $\lambda \in i\mathfrak{a}_{\mathbb{C}}^*$ and we can prove (iii) in the same way as in the proof of Proposition 13.2 (iii). \square

The following is the main result of this section:

Theorem 13.5. *Suppose $\alpha \in \Pi$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfy $\mathbf{e}_{\alpha}(-\lambda) \neq 0$. (Hence both $\tilde{\mathcal{A}}_G(s_{\alpha}, \lambda)$ and $\tilde{\mathcal{A}}_{\mathbf{H}}(s_{\alpha}, \lambda)$ are well defined.) Then*

$$(\tilde{\mathcal{A}}_G(s_{\alpha}, \lambda), \tilde{\mathcal{A}}_{\mathbf{H}}(s_{\alpha}, \lambda)) : (B_G(\lambda)_{K\text{-finite}}, B_{\mathbf{H}}(\lambda)) \longrightarrow (B_G(s_{\alpha} \lambda)_{K\text{-finite}}, B_{\mathbf{H}}(s_{\alpha} \lambda))$$

is a morphism of \mathcal{C}_{rad} . That is, if $V \in \hat{K}_M$ then

(i) for any $\Phi \in \text{Hom}_K(V, C^{\infty}(K/M))$ it holds that

$$(13.7) \quad \tilde{\mathcal{A}}_G(s_{\alpha}, \lambda)(\Phi[v])(1) = \frac{\mathbf{m}_1(\alpha) \Phi[v](1) - \lambda(\alpha^{\vee}) \Phi[s_{\alpha} v](1)}{\mathbf{m}_1(\alpha) - \lambda(\alpha^{\vee})} \quad \forall v \in V_{\text{single}}^M$$

(cf. (13.5)) and

(ii) for any $\Phi \in \text{Hom}_K(V, C^{\infty}(K/M))$ such that

$$\Phi[v](1) = 0 \quad \forall v \in V_{\text{double}}^M$$

it holds that

$$(13.8) \quad \tilde{\mathcal{A}}_G(s_{\alpha}, \lambda)(\Phi[v])(1) = 0 \quad \forall v \in V_{\text{double}}^M.$$

If $\text{Re } \lambda(\alpha^{\vee}) < 0$ then (13.7) is written more explicitly as

$$\int_{\bar{s}_{\alpha}^{-1} N \bar{s}_{\alpha} \cap \theta N} \Phi[s_{\alpha} v](\bar{n}) d\bar{n} = \frac{\mathbf{c}_{\alpha}(-\lambda)}{\mathbf{c}_{\alpha}(\rho)} \cdot \frac{\mathbf{m}_1(\alpha) \Phi[v](1) - \lambda(\alpha^{\vee}) \Phi[s_{\alpha} v](1)}{\mathbf{m}_1(\alpha) - \lambda(\alpha^{\vee})}.$$

Proof. Since each side of (13.7) and (13.8) is holomorphic in λ for any fixed Φ and v , it suffices to prove the theorem when $\lambda \in i\mathfrak{a}_\mathbb{C}^*$. In this case both $\mathcal{P}_G^{s_\alpha\lambda}|_{B_G(s_\alpha\lambda)_{K\text{-finite}}}$ and $\mathcal{P}_\mathbf{H}^{s_\alpha\lambda}$ are bijective. Therefore

$\mathcal{P}^{s_\alpha\lambda} = (\mathcal{P}_G^{s_\alpha\lambda}, \mathcal{P}_\mathbf{H}^{s_\alpha\lambda}) : (B_G(s_\alpha\lambda)_{K\text{-finite}}, B_\mathbf{H}(s_\alpha\lambda)) \longrightarrow (\mathcal{A}(G/K, s_\alpha\lambda)_{K\text{-finite}}, \mathcal{A}(A, s_\alpha\lambda))$ is an isomorphism in \mathcal{C}_{rad} and its inverse is given by

$$(\mathcal{P}^{s_\alpha\lambda})^{-1} = \left((\mathcal{P}_G^{s_\alpha\lambda}|_{B_G(s_\alpha\lambda)_{K\text{-finite}}})^{-1}, (\mathcal{P}_\mathbf{H}^{s_\alpha\lambda})^{-1} \right).$$

Since it follows from Proposition 12.7 that

$$\begin{aligned} \mathcal{P}_G^\lambda|_{B_G(\lambda)_{K\text{-finite}}} &= \mathcal{P}_G^{s_\alpha\lambda}|_{B_G(s_\alpha\lambda)_{K\text{-finite}}} \circ \tilde{\mathcal{A}}_G(s_\alpha, \lambda)|_{B_G(\lambda)_{K\text{-finite}}}, \\ \mathcal{P}_\mathbf{H}^\lambda &= \mathcal{P}_\mathbf{H}^{s_\alpha\lambda} \circ \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, \lambda), \end{aligned}$$

we conclude

$$(\tilde{\mathcal{A}}_G(s_\alpha, \lambda), \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, \lambda)) = (\mathcal{P}^{s_\alpha\lambda})^{-1} \circ \mathcal{P}^\lambda$$

is a morphism in \mathcal{C}_{rad} . □

14. THE HELGASON-FOURIER TRANSFORM AND THE OPDAM-CHEREDNIK TRANSFORM

Helgason introduces the Fourier transform on G/K in [Hel1] as a non-invariant version of the spherical transform while Opdam defines an analogous transform on A in [Op1]. We shall show they constitute a morphism of radial pairs and study some related topics from this view point.

Recall the symbol C_c^∞ stands for the class of compactly supported C^∞ -functions. By Theorem 2.3 for $\mathcal{F} = C_c^\infty$ and Proposition 8.9, $(C_c^\infty(G/K)_{K\text{-finite}}, C_c^\infty(A))$ is a radial pair with radial restriction γ_0 . Put $\ell = \dim \mathfrak{a}$ and let dH be $|W|^{-1}(2\pi)^{-\ell/2}$ times of the Euclidean measure on \mathfrak{a} relative to the metric given by the Killing form $B(\cdot, \cdot)$. Let da be the corresponding Haar measure on A . We normalize a G -invariant non-zero volume element dx on G/K so that

$$(14.1) \quad \int_{G/K} f(x) dx = \int_A f(a) \prod_{\alpha \in \Sigma^+} |2 \sinh \alpha(\log a)|^{\dim \mathfrak{g}_\alpha} da \quad \text{for } f \in C_c^\infty(G/K)^{\ell(K)}$$

(see [Hel4, Ch. I, Theorem 5.8]). The sesquilinear form

$$(f_1, f_2)_{G/K} = \int_{G/K} f_1(x) \overline{f_2(x)} dx$$

on $C_c^\infty(G/K) \times C^\infty(G/K)$ is invariant and non-degenerate and of course the restriction of this form to $C_c^\infty(G/K) \times C_c^\infty(G/K)$ is a Hermitian inner product. Define a sesquilinear form

$$(f_1, f_2)_A = \int_A f_1(a) \overline{f_2(a)} \prod_{\alpha \in \Sigma^+} |2 \sinh \alpha(\log a)|^{\dim \mathfrak{g}_\alpha} da$$

on $C_c^\infty(A) \times C^\infty(A)$. It has the following invariance property:

Proposition 14.1 ([Op1, Lemma 7.8]). *For any $f_1 \in C_c^\infty(A)$, $f_2 \in C^\infty(A)$ and $h \in \mathbf{H}$ it holds that*

$$(\mathcal{T}(h)f_1, f_2)_A = (f_1, \mathcal{T}(h^*)f_2)_A.$$

As in the previous sections, we consider $C^\infty(A)$ (or $C_c^\infty(A)$) as an \mathbf{H} -module by $hf = \mathcal{T}(\theta_{\mathbf{H}}h)f$. Since $(\theta_{\mathbf{H}}h)^* = \theta_{\mathbf{H}}(h^*)$ for $h \in \mathbf{H}$, $(\cdot, \cdot)_A$ is a non-degenerate invariant sesquilinear form on $C_c^\infty(A) \times C^\infty(A)$ which restricts to a Hermitian inner product on $C_c^\infty(A) \times C_c^\infty(A)$. The next proposition is an easy corollary of (14.1):

Proposition 14.2. *Suppose $V \in \widehat{K}_M$ and take bases $\{v_1, \dots, v_m, \dots, v_n\} \subset V$ and $\{v_i^*\} \subset V^*$ as in Proposition 9.10. Then for any $\Phi_1 \in \text{Hom}_K(V, C_c^\infty(G/K)_{K\text{-finite}})$ and $\Phi_2 \in \text{Hom}_K(V^*, C^\infty(G/K)_{K\text{-finite}})$*

$$\sum_{i=1}^n (\Phi_1[v_i], \Phi_2[v_i^*])_{G/K} = \sum_{i=1}^m (\Gamma_0^V(\Phi_1)[v_i], \Gamma_0^{V^*}(\Phi_2)[v_i^*])_A.$$

In particular, for $(C_c^\infty(G/K)_{K\text{-finite}}, C_c^\infty(A))$ and $(C^\infty(G/K)_{K\text{-finite}}, C^\infty(A))$, the pair of $(\cdot, \cdot)_{G/K}$ and $(\cdot, \cdot)_A$ is compatible with restriction in the sense of Definition 9.9.

The transforms we study in this section are the following:

Definition 14.3 (the Helgason-Fourier transform [Hel5, Ch. III, §1, No. 1]). For $g \in G$ let $A(g)$ be as in §12. Suppose $f(x) \in C_c^\infty(G/K)$. For $\lambda \in \mathfrak{a}^*$ and $k \in K$ we put

$$\mathcal{F}_G f(\lambda, k) = \int_{G/K} f(x) e^{(-i\lambda + \rho)(A(k^{-1}x))} dx.$$

Since $\mathcal{F}_G f(\lambda, km) = \mathcal{F}_G f(\lambda, k)$ for any $m \in M$, $\mathcal{F}_G f(\lambda, k)$ is a function on $\mathfrak{a}^* \times K/M$. Since the integral converges for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\mathcal{F}_G f(\lambda, k)$ extends to an analytic function on $\mathfrak{a}_{\mathbb{C}}^* \times K/M$ which is holomorphic in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Definition 14.4 (the Opdam-Cherednik transform [Op1, Definition 7.9]). Suppose $f(a) \in C_c^\infty(A)$. For $\lambda \in \mathfrak{a}^*$ and $w \in W$ we put

$$\mathcal{F}_{\mathbf{H}} f(\lambda, w) = \int_A f(a) \mathbf{G}(-i\lambda, w^{-1}a) \prod_{\alpha \in \Sigma^+} |2 \sinh \alpha(\log a)|^{\dim \mathfrak{g}_\alpha} da.$$

One can see from Theorem 11.2 that for each fixed $w \in W$, $\mathcal{F}_{\mathbf{H}} f(\lambda, w)$ extends to an entire holomorphic function in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Let us pack the target spaces of these transforms into a radial pair. For any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ naturally $B_G(i\lambda) \simeq C^\infty(K/M)$ and $B_{\mathbf{H}}(i\lambda) \simeq (\mathbb{C}W)^*$ by restriction. Using injections

$$\begin{aligned} C^\infty(\mathfrak{a}^* \times K/M) \ni F(\lambda, k) &\longmapsto (k \mapsto F(\lambda, k))_{\lambda \in \mathfrak{a}^*} \in \prod_{\lambda \in \mathfrak{a}^*} B_G(i\lambda), \\ C^\infty(\mathfrak{a}^* \times W) \ni F(\lambda, w) &\longmapsto (w \mapsto F(\lambda, w))_{\lambda \in \mathfrak{a}^*} \in \prod_{\lambda \in \mathfrak{a}^*} B_{\mathbf{H}}(i\lambda), \end{aligned}$$

we can consider $C^\infty(\mathfrak{a}^* \times K/M)$ and $C^\infty(\mathfrak{a}^* \times W)$ as G - and \mathbf{H} -modules respectively. In fact one easily checks that these two spaces are closed under the action of G or \mathbf{H} and that G acts continuously on $C^\infty(\mathfrak{a}^* \times K/M)$ equipped with the topology of compact

convergence in all derivatives. By a similar argument to §12, for each fixed $\lambda \in \mathfrak{a}^*$ the map

$$\begin{aligned} C_c^\infty(G/K) \ni f(x) &\longmapsto \left(G \ni g \longmapsto \int_{G/K} f(x) e^{(-i\lambda+\rho)(A(g^{-1}x))} dx \right. \\ &\quad \left. = (f(x), e^{(i\lambda+\rho)(A(g^{-1}x))})_{G/K} \right) \in B_G(i\lambda) \end{aligned}$$

and the map

$$\begin{aligned} C_c^\infty(A) \ni f(a) &\mapsto \left(\mathbf{H} \ni h \mapsto \int_A f(a) (\mathcal{T}(\theta_{\mathbf{H}}h) \mathbf{G}(-i\lambda, a)) \prod_{\alpha \in \Sigma^+} |2 \sinh \alpha(\log a)|^{\dim \mathfrak{g}_\alpha} da \right. \\ &\quad \left. = (f(a), \mathcal{T}(\overline{\theta_{\mathbf{H}}h}) \mathbf{G}(i\lambda, a))_A \right) \in B_{\mathbf{H}}(i\lambda) \end{aligned}$$

are respectively G - and \mathbf{H} -homomorphisms. This shows \mathcal{F}_G and $\mathcal{F}_{\mathbf{H}}$ are homomorphisms (the continuity of \mathcal{F}_G is clear from the definition). Define the restriction map $\gamma_B : C^\infty(\mathfrak{a}^* \times K/M) \rightarrow C^\infty(\mathfrak{a}^* \times W)$ by

$$F(\lambda, k) \longmapsto (\mathfrak{a}^* \times W \ni (\lambda, w) \longmapsto F(\lambda, \bar{w}) \in \mathbb{C}).$$

Note that $\gamma_B(F)(\lambda, \cdot) = \gamma_{B(i\lambda)}(F(\lambda, \cdot))$ for any $\lambda \in \mathfrak{a}^*$. Suppose $V \in \widehat{K}_M$. If $\Phi \in \text{Hom}_K(V, C^\infty(\mathfrak{a}^* \times K/M))$ then

$$\Gamma_B^V(\Phi) := \gamma_B \circ \Phi|_{V^M}$$

belongs to $\text{Hom}_W(V^M, C^\infty(\mathfrak{a}^* \times W))$. Conversely if $\varphi \in \text{Hom}_W(V^M, C^\infty(\mathfrak{a}^* \times W))$ then $\Phi \in \text{Hom}_K(V, C^\infty(\mathfrak{a}^* \times K/M))$ defined by

$$(14.2) \quad \Phi[v](\lambda, k) = \varphi[p^V(k^{-1}v)](\lambda, 1),$$

with p^V as in Theorem 2.3 (ii), is a unique lift satisfying $\Gamma_B^V(\Phi) = \varphi$. Hence it readily follows from Theorem 9.8 that $(C^\infty(\mathfrak{a}^* \times K/M)_{K\text{-finite}}, C^\infty(\mathfrak{a}^* \times W))$ is a radial pair with radial restriction γ_B .

Remark 14.5. For $w \in W$ let δ_w ($w \in W$) be the element in $B_{\mathbf{H}}(\lambda) \simeq (\mathbb{C}W)^*$ such that

$$\delta_w(t) = \begin{cases} 1 & (t = w), \\ 0 & (t \neq w). \end{cases}$$

Then $B_{\mathbf{H}}(\lambda) = \mathbf{H} \delta_{w_0}$ and by (9.1) one easily sees

$$\xi \delta_{w_0} = -(w_0 \lambda)(\xi) \delta_{w_0} \quad \text{for } \xi \in \mathfrak{a}_{\mathbb{C}}.$$

Hence $B_{\mathbf{H}}(\lambda) \simeq I_{-w_0 \lambda}$ by $\delta_{w_0} \leftrightarrow 1 \otimes 1$ where we put $I_\lambda = \mathbf{H} \otimes_{S(\mathfrak{a}_{\mathbb{C}})} \mathbb{C}_\lambda$ for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ according to [Op1]. On the other hand, if $I_\lambda^{\theta_{\mathbf{H}}}$ denotes I_λ endowed with the twisted \mathbf{H} -module structure by $\theta_{\mathbf{H}}$, then $I_\lambda^{\theta_{\mathbf{H}}} \simeq I_{-w_0 \lambda}$ by $w_0 \otimes 1 \leftrightarrow 1 \otimes 1$. Hence $I_\lambda^{\theta_{\mathbf{H}}} \simeq B_{\mathbf{H}}(\lambda)$ by $w \otimes 1 \leftrightarrow \delta_w$ ($\forall w \in W$).

Now let $\mathcal{F}'_{\mathbf{H}}$ be exactly the same as Opdam's Cherednik transform ' \mathcal{F} ' defined in [Op1, Definition 7.9] and let (\cdot, \cdot) be the inner product on $I_{i\lambda}$ ($\lambda \in \mathfrak{a}^*$) used in his definition. Then for $\lambda \in \mathfrak{a}^*$, $\mathcal{F}'_{\mathbf{H}}(i\lambda) \in I_{i\lambda}$ is defined by

$$\begin{aligned} (\mathcal{F}'_{\mathbf{H}}f(i\lambda), w \otimes 1) &= (2\pi)^{\ell/2}|W| \int_A f(a) \mathbf{G}(-i\lambda, w^{-1}a) da \\ &= (2\pi)^{\ell/2}|W| \mathcal{F}_{\mathbf{H}}f(\lambda, w) \quad \text{for } \forall w \in W \end{aligned}$$

(note Opdam employs $(2\pi)^{\ell/2}|W|da$ as a Haar measure on A) and under the identification $I_{i\lambda} = I_{i\lambda}^{\theta_{\mathbf{H}}} \simeq B_{\mathbf{H}}(i\lambda) \simeq (\mathbb{C}W)^*$ it holds that

$$\mathcal{F}'_{\mathbf{H}}(i\lambda) = (2\pi)^{\ell/2}|W| \mathcal{F}_{\mathbf{H}}f(\lambda, \cdot).$$

We prepare some function spaces.

Definition 14.6. Suppose $\eta > 0$. Let $\text{PW}_{\eta}(\mathfrak{a}^*)$ be the space of holomorphic functions $\psi(\lambda)$ on $\mathfrak{a}_{\mathbb{C}}^*$ such that

$$(14.3) \quad \sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} e^{-\eta|\text{Im } \lambda|} (1 + |\lambda|)^N |\psi(\lambda)| < \infty \quad \text{for each } N \in \mathbb{Z}_{\geq 0}.$$

Let $\text{PW}_{\eta}(\mathfrak{a}^* \times W)$ be the space of holomorphic functions $F(\lambda, w)$ on $\mathfrak{a}_{\mathbb{C}}^* \times W$ such that $F(\cdot, w) \in \text{PW}_{\eta}(\mathfrak{a}^*)$ for each $w \in W$. We naturally consider $\text{PW}_{\eta}(\mathfrak{a}^* \times W) \subset C^{\infty}(\mathfrak{a}^* \times W)$. One easily observes by use of (9.1) that this is an \mathbf{H} -submodule. Let $\text{PW}_{\eta}(\mathfrak{a}^* \times K/M)$ be the space of those continuous functions on $\mathfrak{a}_{\mathbb{C}}^* \times K/M$ which are holomorphic in λ and satisfying

$$(14.4) \quad \sup_{(\lambda, k) \in \mathfrak{a}_{\mathbb{C}}^* \times K/M} e^{-\eta|\text{Im } \lambda|} (1 + |\lambda|)^N |F(\lambda, k)| < \infty \quad \text{for each } N \in \mathbb{Z}_{\geq 0}.$$

This is a Fréchet space by the system of seminorms

$$\|F\|_N = \sup_{(\lambda, k) \in \mathfrak{a}_{\mathbb{C}}^* \times K/M} e^{-\eta|\text{Im } \lambda|} (1 + |\lambda|)^N |F(\lambda, k)| \quad (N \in \mathbb{Z}_{\geq 0}).$$

Moreover we put

$$\text{PW}(\mathfrak{a}^* \times K/M) = \bigcup_{\eta > 0} \text{PW}_{\eta}(\mathfrak{a}^* \times K/M), \quad \text{PW}(\mathfrak{a}^* \times W) = \bigcup_{\eta > 0} \text{PW}_{\eta}(\mathfrak{a}^* \times W).$$

Definition 14.7. Let $\widetilde{\text{PW}}_{\eta}(\mathfrak{a}^* \times K/M)$ be the closed subspace of $\text{PW}_{\eta}(\mathfrak{a}^* \times K/M)$ consisting of those functions F satisfying

$$\begin{aligned} \int_K F(t\lambda, k) e^{(it\lambda + \rho)(A(k^{-1}x))} dk &= \int_K F(\lambda, k) e^{(i\lambda + \rho)(A(k^{-1}x))} dk \\ &\text{for any } t \in W, \lambda \in \mathfrak{a}^* \text{ and } x \in G. \end{aligned}$$

Let $\widetilde{\text{PW}}_{\eta}(\mathfrak{a}^* \times W)$ be the subspace of $\text{PW}_{\eta}(\mathfrak{a}^* \times W)$ consisting of those functions F satisfying

$$\begin{aligned} \sum_{w \in W} F(t\lambda, w) \mathbf{G}(it\lambda, w^{-1}a) &= \sum_{w \in W} F(\lambda, w) \mathbf{G}(i\lambda, w^{-1}a) \\ &\text{for any } t \in W, \lambda \in \mathfrak{a}^* \text{ and } a \in A. \end{aligned}$$

Moreover we put

$$\widetilde{\text{PW}}(\mathfrak{a}^* \times K/M) = \bigcup_{\eta>0} \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M), \quad \widetilde{\text{PW}}(\mathfrak{a}^* \times W) = \bigcup_{\eta>0} \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W).$$

We equip $\widetilde{\text{PW}}(\mathfrak{a}^* \times K/M)$ with the topology of the inductive limit.

Definition 14.8. Let d denote the distance function on G/K or A given by the Riemannian metric corresponding to the Killing form. Put for each $\eta > 0$

$$C_\eta^\infty(G/K) = \{f(x) \in C^\infty(G/K); f(x) = 0 \text{ whenever } d(x, 1K) \geq \eta\},$$

$$C_\eta^\infty(A) = \{f(a) \in C^\infty(A); f(a) = 0 \text{ whenever } d(a, 1) \geq \eta\}.$$

They are Fréchet spaces with the topology of uniform convergence in all derivatives. (The topology of $C_\eta^\infty(A)$ is not used in the paper.)

Now we describe Helgason's results on \mathcal{F}_G with some subsidiary information. Let $d\lambda$ is $|W|^{-1}(2\pi)^{-\ell/2}$ times of the Euclidean measure on \mathfrak{a}^* induced by the Killing form.

Proposition 14.9. (i) If $F \in \text{PW}_\eta(\mathfrak{a}^* \times K/M)$ then

$$(14.5) \quad \mathcal{J}_G F(x) = \int_{\mathfrak{a}^*} \left(\int_K e^{(i\lambda + \rho)(A(k^{-1}x))} F(\lambda, k) dk \right) |\mathbf{c}(\lambda)|^{-2} d\lambda$$

converges for all $x \in G$ and defines a C^∞ function on G/K . Here $\mathbf{c}(\lambda)$ is Harish-Chandra's \mathbf{c} -function defined by

$$\mathbf{c}(\lambda) = \prod_{\alpha \in R_1^+} \frac{\mathbf{c}_\alpha(i\lambda)}{\mathbf{c}_\alpha(\rho)}.$$

The linear map $\mathcal{J}_G : \text{PW}_\eta(\mathfrak{a}^* \times K/M) \rightarrow C^\infty(G/K)$ is continuous ($C^\infty(G/K)$ has the topology of compact convergence in all derivatives).

(ii) The transform \mathcal{F}_G is a bijection of $C_c^\infty(G/K)$ onto $\widetilde{\text{PW}}(\mathfrak{a}^* \times K/M)$. More precisely, for each $\eta > 0$, $C_\eta^\infty(G/K)$ is isomorphic to $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)$ by \mathcal{F}_G as a topological vector space. The inverse map is given by \mathcal{J}_G .

(iii) For any $f_1, f_2 \in C_c^\infty(G/K)$

$$(f_1, f_2)_{G/K} = \int_{\mathfrak{a}^*} \int_K \mathcal{F}_G f_1(\lambda, k) \overline{\mathcal{F}_G f_2(\lambda, k)} dk |\mathbf{c}(\lambda)|^{-2} d\lambda.$$

Corollary 14.10. Any function in $\widetilde{\text{PW}}(\mathfrak{a}^* \times K/M) = \mathcal{F}_G(C_c^\infty(G/K))$ is necessarily analytic in all variables. Hence an embedding $\widetilde{\text{PW}}(\mathfrak{a}^* \times K/M) \hookrightarrow C^\infty(\mathfrak{a}^* \times K/M)$ is naturally defined and this is continuous since $\mathcal{F}_G : C_c^\infty(G/K) \rightarrow C^\infty(\mathfrak{a}^* \times K/M)$ is continuous. Consider $\widetilde{\text{PW}}(\mathfrak{a}^* \times K/M)$ as a G -submodule of $C^\infty(\mathfrak{a}^* \times K/M)$. Since any function in $C_c^\infty(G/K)$ is a C^∞ vector, the same thing holds for $\widetilde{\text{PW}}(\mathfrak{a}^* \times K/M)$. Thus $U(\mathfrak{g}_\mathbb{C})$ acts on $\widetilde{\text{PW}}(\mathfrak{a}^* \times K/M)$ in a compatible way with the embedding. For each $\eta > 0$, $C_\eta^\infty(\mathfrak{a}^* \times K/M)$ and $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)$ are stable under the actions of K and $U(\mathfrak{g}_\mathbb{C})$. Hence we can consider $(\mathfrak{g}_\mathbb{C}, K)$ -modules $C_\eta^\infty(\mathfrak{a}^* \times K/M)_{K\text{-finite}}$ and $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)_{K\text{-finite}}$. They are isomorphic via \mathcal{F}_G or \mathcal{J}_G .

Proof of Proposition 14.9. In the proof of (i) we regard $C^\infty(G/K)$ as a closed subspace of $C^\infty(G)$. Suppose $D \in U(\mathfrak{g}_\mathbb{C})$. Let E be a finite-dimensional $\text{Ad}(K)$ -stable subspace of $U(\mathfrak{g}_\mathbb{C})$ containing D and take a basis $\{D_j\}$ of E . Then there exist analytic functions $\pi_j(k)$ on K such that

$$\text{Ad}(k)D = \sum_j \pi_j(k) D_j \quad \text{for any } k \in K.$$

For $g \in G$ let $\kappa(g)$ denote the unique element in K such that $g = n \exp A(g) \kappa(g)$ for some $n \in N$. For $k \in K$ and $x \in G$ it holds that

$$\begin{aligned} r_x(D) e^{(i\lambda+\rho)(A(k^{-1}x))} &= e^{(i\lambda+\rho)(A(k^{-1}x))} (r_y(\text{Ad}(\kappa(k^{-1}x))D) e^{(i\lambda+\rho)(A(y))}) \Big|_{y=1} \\ &= e^{(i\lambda+\rho)(A(k^{-1}x))} \sum_j \pi_j(\kappa(k^{-1}x)) (r_y(D_j) e^{(i\lambda+\rho)(A(y))}) \Big|_{y=1} \\ &= e^{(i\lambda+\rho)(A(k^{-1}x))} \sum_j \pi_j(\kappa(k^{-1}x)) \gamma(D_j)(i\lambda). \end{aligned}$$

Hence by the inequality $|A(k^{-1}x)| \leq d(xK, 1K)$ (cf. [Hel4, Ch. IV, §10, (14)]) we get for any $F \in \text{PW}_\eta(\mathfrak{a}^* \times K/M)$ and any $N \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} &\sup_{\lambda \in \mathfrak{a}_\mathbb{C}} (1 + |\lambda|)^N \left| r_x(D) \int_K e^{(i\lambda+\rho)(A(k^{-1}x))} F(\lambda, k) dk \right| \\ &\leq \left(\sup_{\substack{H \in \mathfrak{a}; \\ |H| \leq d(xK, 1K)}} e^{\rho(H)} \right) \sum_j \|\pi_j(\cdot)\|_{L^\infty(K)} \sup_{\substack{\lambda \in \mathfrak{a}_\mathbb{C} \\ k \in K}} \left((1 + |\lambda|)^N |\gamma(D_j)(i\lambda) F(\lambda, k)| \right). \end{aligned}$$

Since $|\mathbf{c}(\lambda)|^{-2}$ has at most polynomial growth (cf. [Hel4, Ch. IV, Proposition 7.2]), we can differentiate (14.5) as a function in $x \in G$ repeatedly under the outer integral. The first assertion is thus proved. The continuity of \mathcal{J}_G also easily follows from the above estimate.

To prove (ii) let $F \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)$ be given. We assert there exists $f \in C^\infty_\eta(G/K)$ such that $F = \mathcal{F}_G f$. In fact, if F has C^∞ regularity then the proof for this assertion is given in [Hel5, pp.268–271]. But the proof there works for any continuous F with the help of (i). Thus the bijectivity of

$$(14.6) \quad \mathcal{J}_G : \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M) \rightarrow C^\infty_\eta(G/K)$$

follows from [Hel5, Ch. III, Theorem 5.1]. Since the subspace $C^\infty_\eta(G/K) \subset C^\infty(G/K)$ is a Fréchet space, from (i) and the open mapping theorem one sees (14.6) is an isomorphism of topological vector spaces.

For (iii) we refer the reader to [Hel5, Ch. III, §1, No. 2]. □

Let us return to the argument on the target spaces.

Lemma 14.11. *For $F \in \text{PW}_\eta(\mathfrak{a}^* \times K/M)$*

$$F \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M) \Leftrightarrow \begin{cases} F \in C^\infty(\mathfrak{a}^* \times K/M), \\ \mathcal{P}_G^{it\lambda}(F(t\lambda, \cdot)) = \mathcal{P}_G^{i\lambda}(F(\lambda, \cdot)) \text{ for any } t \in W \text{ and } \lambda \in \mathfrak{a}^* \end{cases}$$

$$\Leftrightarrow \begin{cases} F \in C^\infty(\mathfrak{a}^* \times K/M), \\ \tilde{\mathcal{A}}_G(t, i\lambda)(F(\lambda, \cdot)) = F(t\lambda, \cdot) \text{ for any } t \in W \text{ and } \lambda \in \mathfrak{a}^*. \end{cases}$$

For $F \in \text{PW}_\eta(\mathfrak{a}^* \times W)$

$$\begin{aligned} F \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W) &\Leftrightarrow \mathcal{P}_\mathbf{H}^{it\lambda}(F(t\lambda, \cdot)) = \mathcal{P}_\mathbf{H}^{i\lambda}(F(\lambda, \cdot)) \text{ for any } t \in W \text{ and } \lambda \in \mathfrak{a}^* \\ &\Leftrightarrow \tilde{\mathcal{A}}_\mathbf{H}(t, i\lambda)(F(\lambda, \cdot)) = F(t\lambda, \cdot) \text{ for any } t \in W \text{ and } \lambda \in \mathfrak{a}^*. \end{aligned}$$

In particular $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$ is an \mathbf{H} -submodule of $\text{PW}_\eta(\mathfrak{a}^* \times W)$.

Proof. Suppose $F \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$. Then $F \in C^\infty(\mathfrak{a}^* \times K/M)$ by Corollary 14.10 and for any $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we can consider $F(\lambda, \cdot) \in B_G(i\lambda)$ to which $\mathcal{P}_G^{i\lambda}$ and $\tilde{\mathcal{A}}_G(t, i\lambda)$ can be applied. Hence the lemma is immediate from Definition 14.7, the definition of the Poisson transforms, and the identities

$$\mathcal{P}_G^{i\lambda} = \mathcal{P}_G^{it\lambda} \circ \tilde{\mathcal{A}}_G(t, i\lambda), \quad \mathcal{P}_\mathbf{H}^{i\lambda} = \mathcal{P}_\mathbf{H}^{it\lambda} \circ \tilde{\mathcal{A}}_\mathbf{H}(t, i\lambda). \quad \square$$

Proposition 14.12. *For any $\eta > 0$ the pair $(C_\eta^\infty(G/K)_{K\text{-finite}}, C_\eta^\infty(A))$ is a radial pair with radial restriction γ_0 and is a subobject of $(C_c^\infty(G/K)_{K\text{-finite}}, C_c^\infty(A))$. Likewise $(\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)_{K\text{-finite}}, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W))$ is a subobject of $(C^\infty(\mathfrak{a}^* \times K/M)_{K\text{-finite}}, C^\infty(\mathfrak{a}^* \times W)) \in \mathcal{C}_{\text{rad}}$.*

Proof. First, it is clear that $\gamma_0(C_\eta^\infty(G/K)) \subset C_\eta^\infty(A)$. Suppose $V \in \hat{K}_M$. For any $\varphi \in \text{Hom}_W(V_{\text{single}}^M, C_\eta^\infty(A))$ let Φ be its lift in $\text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(G/K))$. Extending φ to an element of $\text{Hom}_W(V^M, C_\eta^\infty(A))$ by $\varphi|_{V_{\text{double}}^M} = 0$, we have

$$\Phi[v](kak^{-1}) = \varphi[p^V(k^{-1}v)](a) \quad \text{for any } v \in V, k \in K \text{ and } a \in A.$$

From this we easily see $\Phi \in \text{Hom}_K(V, C_\eta^\infty(G/K))$. These facts prove the first assertion.

Let us prove the second assertion. If $F \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)$ then $\gamma_B(F) \in \text{PW}_\eta(\mathfrak{a}^* \times W)$ since Condition (14.4) for F implies Condition (14.3) for each $\gamma_B(F)(\cdot, w)$ ($w \in W$). Suppose $V \in \hat{K}_M$. For any $\Phi \in \text{Hom}_K(V, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M))$ put

$$\tilde{\Gamma}_B^V(\Phi) = \gamma_B \circ \Phi|_{V_{\text{single}}^M} \in \text{Hom}_W(V_{\text{single}}^M, \text{PW}_\eta(\mathfrak{a}^* \times W)).$$

Then for any $\lambda \in \mathfrak{a}^*$ we have

$$(14.7) \quad \Phi_\lambda := (V \ni v \mapsto \Phi[v](\lambda, \cdot)) \in \text{Hom}_K(V, B_G(i\lambda)),$$

$$\tilde{\Gamma}_B^V(\Phi)_\lambda := (V_{\text{single}}^M \ni v \mapsto \tilde{\Gamma}_B^V(\Phi)[v](\lambda, \cdot)) \in \text{Hom}_W(V_{\text{single}}^M, B_\mathbf{H}(i\lambda)),$$

$$(14.8) \quad \tilde{\Gamma}_{B(i\lambda)}^V(\Phi_\lambda) := \gamma_{B(i\lambda)} \circ \Phi_\lambda|_{V_{\text{single}}^M} = \tilde{\Gamma}_B^V(\Phi)_\lambda.$$

Now for any $t \in W$

$$\begin{aligned} \tilde{\mathcal{A}}_\mathbf{H}(t, i\lambda) \circ \tilde{\Gamma}_B^V(\Phi)_\lambda &= \tilde{\mathcal{A}}_\mathbf{H}(t, i\lambda) \circ \tilde{\Gamma}_{B(i\lambda)}^V(\Phi_\lambda) & (\because (14.8)) \\ &= \tilde{\Gamma}_{B(it\lambda)}^V(\tilde{\mathcal{A}}_G(t, i\lambda) \circ \Phi_\lambda) & (\because \text{Theorem 13.5}) \\ &= \tilde{\Gamma}_{B(it\lambda)}^V(\Phi_{t\lambda}) & (\because \text{Lemma 14.11}) \\ &= \tilde{\Gamma}_B^V(\Phi)_{t\lambda}. & (\because (14.8)) \end{aligned}$$

Hence by Lemma 14.11 we conclude $\tilde{\Gamma}_B^V(\Phi) \in \text{Hom}_W(V_{\text{single}}^M, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W))$, namely

$$(14.9) \quad \tilde{\Gamma}_B^V(\text{Hom}_K(V, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M))) \subset \text{Hom}_W(V_{\text{single}}^M, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)).$$

Conversely, suppose $\varphi \in \text{Hom}_W(V_{\text{single}}^M, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W))$ and let Φ be its unique lift in $\text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{a}^* \times K/M))$. We extend φ to an element of $\text{Hom}_W(V^M, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W))$ by $\varphi|_{V_{\text{double}}^M} = 0$. Then (14.2) holds for any $v \in V$, $k \in K$ and $\lambda \in \mathfrak{a}^*$. This means for each $v \in V$, $\Phi[v](\lambda, k)$ extends to an analytic function on $\mathfrak{a}_{\mathbb{C}}^* \times K/M$ which is holomorphic in λ . Let $|\cdot|$ be the norm of V induced from a K -invariant inner product. Then for each $N \in \mathbb{Z}_{\geq 0}$ there exists $C_N > 0$ such that

$$\sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} e^{-\eta|\text{Im } \lambda|} (1 + |\lambda|)^N |\varphi[v](\lambda, 1)| < C_N \quad \text{for any } v \in V^M \text{ with } |v| \leq 1.$$

Since $|p^V(k^{-1}v)| \leq |v|$ for any $k \in K$ and $v \in V$, $F := \Phi[v]$ satisfies (14.4) for $v \in V$ with $|v| \leq 1$. Thus $\Phi[v] \in \text{PW}(\mathfrak{a}^* \times K/M)$ for all $v \in V$. Now for any $\lambda \in \mathfrak{a}^*$ let Φ_λ be as in (14.7) and put

$$\varphi_\lambda := (V_{\text{single}}^M \ni v \mapsto \varphi[v](\lambda, \cdot)) \in \text{Hom}_W(V_{\text{single}}^M, B_{\mathbf{H}}(i\lambda)).$$

Then $\Phi_\lambda \in \text{Hom}_K^{2 \rightarrow 2}(V, B_G(i\lambda))$ and for any $t \in W$ we have

$$\begin{aligned} \Phi_{t\lambda} &\in \text{Hom}_K^{2 \rightarrow 2}(V, B_G(it\lambda)), \\ \tilde{\mathcal{A}}_G(t, i\lambda) \circ \Phi_\lambda &\in \text{Hom}_K^{2 \rightarrow 2}(V, B_G(it\lambda)), & (\because \text{Theorem 13.5}) \\ \tilde{\Gamma}_{B(it\lambda)}^V(\tilde{\mathcal{A}}_G(t, i\lambda) \circ \Phi_\lambda) &= \tilde{\mathcal{A}}_{\mathbf{H}}(t, i\lambda) \circ \tilde{\Gamma}_{B(i\lambda)}^V(\Phi_\lambda) & (\because \text{Theorem 13.5}) \\ &= \tilde{\mathcal{A}}_{\mathbf{H}}(t, i\lambda) \circ \varphi_\lambda \\ &= \varphi_{t\lambda} & (\because \text{Lemma 14.11}) \\ &= \tilde{\Gamma}_{B(it\lambda)}^V(\Phi_{t\lambda}) \end{aligned}$$

and hence $\tilde{\mathcal{A}}_G(t, i\lambda) \circ \Phi_\lambda = \Phi_{t\lambda}$. Hence $\Phi \in \text{Hom}_K(V, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M))$ by Lemma 14.11. Thus $\tilde{\Gamma}_B^V$ induces a linear bijection

$$\begin{aligned} \tilde{\Gamma}_B^V : \text{Hom}_K(V, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)) \cap \text{Hom}_K^{2 \rightarrow 2}(V, C^\infty(\mathfrak{a}^* \times K/M)) \\ \simeq \text{Hom}_W(V_{\text{single}}^M, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)). \end{aligned}$$

This and (14.9) prove that $(\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)_{K\text{-finite}}, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W))$ is a subobject of $(C^\infty(\mathfrak{a}^* \times K/M)_{K\text{-finite}}, C^\infty(\mathfrak{a}^* \times W))$ in \mathcal{C}_{Ch} (and hence in \mathcal{C}_{rad}). \square

To import Opdam's result on the characterization of the image of $\mathcal{F}_{\mathbf{H}}$ we shall prepare another description of $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$. Put $\mathcal{M} = S(\mathfrak{a}_{\mathbb{C}})\mathfrak{a}_{\mathbb{C}}$ and let $\hat{S}(\mathfrak{a}_{\mathbb{C}})$ be the \mathcal{M} -adic completion of $S(\mathfrak{a}_{\mathbb{C}})$, namely the algebra of formal power series at $0 \in \mathfrak{a}_{\mathbb{C}}^*$. Then there exists uniquely an algebra $\hat{\mathbf{H}}$ over \mathbb{C} with the following properties:

- (i) $\hat{\mathbf{H}} \simeq \hat{S}(\mathfrak{a}_{\mathbb{C}}) \otimes \mathbb{C}W$ as a \mathbb{C} -linear space;
- (ii) The maps $\hat{S}(\mathfrak{a}_{\mathbb{C}}) \rightarrow \hat{\mathbf{H}}, \psi \mapsto \psi \otimes 1$ and $\mathbb{C}W \rightarrow \hat{\mathbf{H}}, w \mapsto 1 \otimes w$ are algebra homomorphisms;
- (iii) $(\psi \otimes 1) \cdot (1 \otimes w) = \psi \otimes w$ for any $\psi \in \hat{S}(\mathfrak{a}_{\mathbb{C}})$ and $w \in W$;

(iv) $(1 \otimes s_\alpha) \cdot (\psi(\lambda) \otimes 1) = \psi(s_\alpha \lambda) \otimes s_\alpha - \mathbf{k}(\alpha) \frac{\psi(\lambda) - \psi(s_\alpha \lambda)}{\alpha^\vee}$ for any $\alpha \in \Pi$ and $\psi \in \widehat{S}(\mathfrak{a}_\mathbb{C})$.

We identify \mathbf{H} , $\widehat{S}(\mathfrak{a}_\mathbb{C})$ and $\mathbb{C}W$ with subalgebras of $\widehat{\mathbf{H}}$ in obvious ways. By (iv) we have the following:

Lemma 14.13. *For any $\psi \in \widehat{S}(\mathfrak{a}_\mathbb{C})$ and $w \in W$ define $\psi_t^w \in \widehat{S}(\mathfrak{a}_\mathbb{C})$ ($t \in W$) by the identity*

$$w^{-1}\psi = \sum_{t \in W} \psi_t^w \otimes t^{-1}$$

in $\widehat{\mathbf{H}}$. Then for each $w, t \in W$ the correspondence $\psi \mapsto \psi_t^w$ is continuous with respect to the \mathcal{M} -adic topology.

For each $\eta > 0$ we identify the function space

$$\text{PW}_\eta(-i\mathfrak{a}^*) := \{\psi(i\lambda); \psi \in \text{PW}_\eta(\mathfrak{a}^*)\}$$

with a subspace of $\widehat{S}(\mathfrak{a}_\mathbb{C})$. Then it holds that

$$(14.10) \quad \text{PW}_\eta(-i\mathfrak{a}^*) \cdot W = W \cdot \text{PW}_\eta(-i\mathfrak{a}^*) = \mathbf{H} \cdot \text{PW}_\eta(-i\mathfrak{a}^*)$$

in $\widehat{\mathbf{H}}$ (cf. [Op1, §8]). Let

$$\mathbf{1}_\mathbf{H}(\lambda, w) \in C^\infty(\mathfrak{a}^* \times W) \subset \prod_{\lambda \in \mathfrak{a}^*} B_\mathbf{H}(i\lambda)$$

be the constant function with value 1. Let $\pi_\mathbf{H}$ denote the action of \mathbf{H} on $\prod_{\lambda \in \mathfrak{a}^*} B_\mathbf{H}(i\lambda)$. For any $\psi \in \widehat{S}(\mathfrak{a}_\mathbb{C})$ take a sequence $\{\psi_n\} \subset S(\mathfrak{a}_\mathbb{C})$ converging to ψ with respect to the \mathcal{M} -adic topology. Then for each $w \in W$ $\{(\pi_\mathbf{H}(\psi_n)\mathbf{1}_\mathbf{H})(\cdot, w)\} \subset S(\mathfrak{a}_\mathbb{C})$ converges to an element of $\widehat{S}(\mathfrak{a}_\mathbb{C})$, the limit being independent of the choice of sequences. More precisely, if we take $\psi_t^w \in \widehat{S}(\mathfrak{a}_\mathbb{C})$ ($t \in W$) for ψ as in Lemma 14.13, then

$$(14.11) \quad \left(\lim_{n \rightarrow \infty} (\pi_\mathbf{H}(\psi_n)\mathbf{1}_\mathbf{H})(\cdot, w) \right)(\lambda) = \sum_{t \in W} \psi_t^w(-i\lambda).$$

Hence because of (14.10), if $\psi \in \text{PW}_\eta(-i\mathfrak{a}^*)$ then we can define a function $\pi_\mathbf{H}(\psi)\mathbf{1}_\mathbf{H} \in \text{PW}_\eta(\mathfrak{a}^* \times W)$ by

$$(\pi_\mathbf{H}(\psi)\mathbf{1}_\mathbf{H})(\lambda, w) = \left(\lim_{n \rightarrow \infty} (\pi_\mathbf{H}(\psi_n)\mathbf{1}_\mathbf{H})(\cdot, w) \right)(\lambda).$$

Using (14.10) again, we see $\text{PW}_\eta(-i\mathfrak{a}^*) \otimes \mathbb{C}_{\text{triv}}$ is an \mathbf{H} -submodule of $\widehat{S}(\mathfrak{a}_\mathbb{C}) \otimes \mathbb{C}_{\text{triv}} \simeq \widehat{\mathbf{H}} \otimes_{\mathbb{C}W} \mathbb{C}_{\text{triv}}$. Now fix a $v_{\text{triv}} \in \mathbb{C}_{\text{triv}} \setminus \{0\}$. It is then easy to see that the linear map

$$\mathcal{L}_\mathbf{H} : \text{PW}_\eta(-i\mathfrak{a}^*) \otimes \mathbb{C}_{\text{triv}} \longrightarrow \text{PW}_\eta(\mathfrak{a}^* \times W); \quad \psi \otimes v_{\text{triv}} \longmapsto \pi_\mathbf{H}(\psi)\mathbf{1}_\mathbf{H}$$

is an \mathbf{H} -homomorphism.

Proposition 14.14. *The homomorphism $\mathcal{L}_\mathbf{H}$ is a bijection of $\text{PW}_\eta(-i\mathfrak{a}^*) \otimes \mathbb{C}_{\text{triv}}$ onto $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$. If $F(\lambda, w) \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$ then $\mathcal{L}_\mathbf{H}^{-1}F(\lambda) = F(i\lambda, 1) \otimes v_{\text{triv}}$.*

Proof. Suppose $\psi \in \text{PW}_\eta(-i\mathfrak{a}^*)$ and let us prove $\mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}}) \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$. Let $\{\psi_n\} \subset S(\mathfrak{a}_\mathbb{C})$ be the sequence converging to ψ and put $F_n(\lambda, w) = (\pi_\mathbf{H}(\psi_n)\mathbf{1}_\mathbf{H})(\lambda, w) \in C^\infty(\mathfrak{a}^* \times W)$. Then it follows from (13.5) that for each $\alpha \in \Pi$, $w \in W$ and $\lambda \in \mathfrak{a}^*$

$$\tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, i\lambda)(F_n(\lambda, \cdot))(w) = \frac{\mathbf{m}_1(\alpha)F_n(\lambda, w) - i\lambda(\alpha^\vee)F_n(\lambda, ws_\alpha)}{\mathbf{m}_1(\alpha) - i\lambda(\alpha^\vee)}.$$

For fixed α and w , the right-hand side converges to

$$\begin{aligned} & \frac{\mathbf{m}_1(\alpha)(\pi_\mathbf{H}(\psi)\mathbf{1}_\mathbf{H})(\lambda, w) - i\lambda(\alpha^\vee)(\pi_\mathbf{H}(\psi)\mathbf{1}_\mathbf{H})(\lambda, ws_\alpha)}{\mathbf{m}_1(\alpha) - i\lambda(\alpha^\vee)} \\ &= \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, i\lambda)((\pi_\mathbf{H}(\psi)\mathbf{1}_\mathbf{H})(\lambda, \cdot))(w) = \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, i\lambda)(\mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}})(\lambda, \cdot))(w) \end{aligned}$$

in $\widehat{S}(\mathfrak{a}_\mathbb{C})$ while the left-hand side equals

$$\begin{aligned} \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, i\lambda)((\pi_\mathbf{H}(\psi_n)\mathbf{1}_\mathbf{H})(\lambda, \cdot))(w) &= \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, i\lambda)(\psi_n \mathbf{1}_\mathbf{H}^{i\lambda})(w) \\ &= (\psi_n \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, i\lambda)\mathbf{1}_\mathbf{H}^{i\lambda})(w) \\ &= (\psi_n \mathbf{1}_\mathbf{H}^{is_\alpha\lambda})(w) \quad (\because (13.4)) \\ &= (\pi_\mathbf{H}(\psi_n)\mathbf{1}_\mathbf{H})(s_\alpha\lambda, w) \end{aligned}$$

and converges to $\mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}})(s_\alpha\lambda, w)$ in $\widehat{S}(\mathfrak{a}_\mathbb{C})$. This shows

$$\tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, i\lambda)(\mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}})(\lambda, \cdot)) = \mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}})(s_\alpha\lambda, \cdot) \quad \text{for } \alpha \in \Pi \text{ and } \lambda \in \mathfrak{a}^*.$$

Hence from Lemma 14.11 we conclude $\mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}}) \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$.

Secondly, for a given $\psi \in \text{PW}_\eta(-i\mathfrak{a}^*)$ let $\psi_t^1 \in \widehat{S}(\mathfrak{a}_\mathbb{C})$ ($t \in W$) be as in Lemma 14.13 for $w = 1$. Then

$$\psi_t^1 = \begin{cases} \psi & \text{if } t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence by (14.11) we have $\mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}})(\lambda, 1) = \psi(-i\lambda)$. This proves the injectivity of $\mathcal{L}_\mathbf{H}$.

In order to show the surjectivity, let $F \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$ be given. We put $\psi(\lambda) = F(i\lambda, 1)$. Then $\psi \in \text{PW}_\eta(-i\mathfrak{a}^*)$ and $F' := F - \mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}})$ satisfies $F'(\lambda, 1) = 0$. We assert $F' = 0$. Indeed, since $F' \in \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$, it follows from Lemma 14.11 and (13.5) that for any $\alpha \in \Pi$ and $w \in W$

$$F'(s_\alpha\lambda, w) = \tilde{\mathcal{A}}_\mathbf{H}(s_\alpha, i\lambda)(F'(\lambda, \cdot))(w) = \frac{\mathbf{m}_1(\alpha)F'(\lambda, w) - i\lambda(\alpha^\vee)F'(\lambda, ws_\alpha)}{\mathbf{m}_1(\alpha) - i\lambda(\alpha^\vee)}.$$

This means $F'(\cdot, w) = 0$ implies $F'(\cdot, ws_\alpha) = 0$. Hence we can prove $F'(\cdot, w) = 0$ for all $w \in W$ by the induction in the length of w . Thus $F = \mathcal{L}_\mathbf{H}(\psi \otimes v_{\text{triv}})$. \square

Now we are in the position of stating an analogue of Proposition 14.9 for \mathbf{H} .

Proposition 14.15. (i) *If $F \in \text{PW}_\eta(\mathfrak{a}^* \times W)$ then*

$$(14.12) \quad \mathcal{J}_\mathbf{H}F(a) = \int_{\mathfrak{a}^*} \frac{1}{|W|} \sum_{w \in W} \mathbf{G}(i\lambda, w^{-1}a) F(\lambda, w) |\mathbf{c}(\lambda)|^{-2} d\lambda.$$

absolutely converges for all $a \in A$ and defines a C^∞ function on A . The linear map $\mathcal{J}_{\mathbf{H}} : \text{PW}_\eta(\mathfrak{a}^* \times W) \rightarrow C^\infty(A)$ is an \mathbf{H} -homomorphism.

(ii) The transform $\mathcal{F}_{\mathbf{H}}$ is a bijection of $C_c^\infty(A)$ onto $\widetilde{\text{PW}}(\mathfrak{a}^* \times W)$. More precisely, for each $\eta > 0$, $C_\eta^\infty(A)$ is isomorphic to $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$ by $\mathcal{F}_{\mathbf{H}}$ as an \mathbf{H} -module. The inverse map is given by $\mathcal{J}_{\mathbf{H}}$.

(iii) For any $f_1, f_2 \in C_c^\infty(A)$

$$(f_1, f_2)_A = \int_{\mathfrak{a}^*} \frac{1}{|W|} \sum_{w \in W} \mathcal{F}_{\mathbf{H}} f_1(\lambda, w) \overline{\mathcal{F}_{\mathbf{H}} f_2(\lambda, w)} |\mathbf{c}(\lambda)|^{-2} d\lambda.$$

Proof. Thanks to the estimate of \mathbf{G} in [Op1, Corollary 6.2], the integral (14.12) converges and can be differentiated repeatedly in a under the integral sign. Hence for any $h \in \mathbf{H}$ and any regular $a \in A$

$$\begin{aligned} \mathcal{T}(\theta_{\mathbf{H}} h) \mathcal{J}_{\mathbf{H}} F(a) &= \int_{\mathfrak{a}^*} \mathcal{T}(\theta_{\mathbf{H}} h) \left(\frac{1}{|W|} \sum_{w \in W} \mathbf{G}(i\lambda, w^{-1} \cdot) F(\lambda, w) \right) (a) |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= \int_{\mathfrak{a}^*} \mathcal{T}(\theta_{\mathbf{H}} h) (\mathcal{P}_{\mathbf{H}}^{i\lambda}(F(\lambda, \cdot))) (a) |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= \int_{\mathfrak{a}^*} \mathcal{P}_{\mathbf{H}}^{i\lambda}(hF(\lambda, \cdot)) (a) |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= \int_{\mathfrak{a}^*} \mathcal{P}_{\mathbf{H}}^{i\lambda}((\pi_{\mathbf{H}}(h)F)(\lambda, \cdot)) (a) |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= \mathcal{J}_{\mathbf{H}}(\pi_{\mathbf{H}}(h)F)(a). \end{aligned}$$

Thus we get (i).

Opdam shows in [Op1, §§8, 9] that $\mathcal{F}_{\mathbf{H}}$ gives a linear bijection of $C_\eta^\infty(A)$ onto $\mathcal{L}_{\mathbf{H}}(\text{PW}_\eta(-i\mathfrak{a}^*) \otimes \mathbb{C}_{\text{triv}})$ and the inverse map is given by

$$\mathcal{J}'_{\mathbf{H}} F(a) := |W|^2 \left(\prod_{\alpha \in R_1^+} \frac{\rho(\alpha^\vee)}{\rho(\alpha^\vee) + \mathbf{m}_1(\alpha)} \right)^2 \int_{\mathfrak{a}_+^*} \sum_{w \in W} \mathbf{G}(i\lambda, w^{-1}a) F(\lambda, w) |\mathbf{c}(\lambda)|^{-2} d\lambda$$

where $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^*; \lambda(\alpha^\vee) > 0 \text{ for all } \alpha \in \Pi\}$. (Although he uses another kind of support conditions, his proof works in our setting.) By Proposition 14.14, (ii) follows if we can prove $\mathcal{J}'_{\mathbf{H}} F = \mathcal{J}_{\mathbf{H}} F$ for $F \in \widetilde{\text{PW}}(\mathfrak{a}^* \times W)$. First, from Definition 14.7 and the W -invariance of $|\mathbf{c}(\lambda)|^{-2}$ we have

$$\int_{\mathfrak{a}_+^*} \sum_{w \in W} \mathbf{G}(i\lambda, w^{-1}a) F(\lambda, w) |\mathbf{c}(\lambda)|^{-2} d\lambda = \frac{1}{|W|} \int_{\mathfrak{a}^*} \sum_{w \in W} \mathbf{G}(i\lambda, w^{-1}a) F(\lambda, w) |\mathbf{c}(\lambda)|^{-2} d\lambda.$$

Secondly, by [Op1, Proposition 1.4 (3)] it holds that for any $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $a \in A$

$$\left(\prod_{\alpha \in R_1^+} \lambda(\alpha^\vee) \right) \sum_{w \in W} \mathbf{G}(\lambda, wa) = \sum_{w \in W} (\text{sgn } w) \left(\prod_{\alpha \in R_1^+} ((w\lambda)(\alpha^\vee) - \mathbf{m}_1(\alpha)) \right) \mathbf{G}(w\lambda, a).$$

Specializing this to $(\lambda, a) = (-\rho, 1)$, we get

$$|W| \prod_{\alpha \in R_1^+} \rho(\alpha^\vee) = \prod_{\alpha \in R_1^+} (\rho(\alpha^\vee) + \mathbf{m}_1(\alpha))$$

and hence

$$|W|^2 \left(\prod_{\alpha \in R_1^+} \frac{\rho(\alpha^\vee)}{\rho(\alpha^\vee) + \mathbf{m}_1(\alpha)} \right)^2 = 1.$$

Thus $\mathcal{J}'_{\mathbf{H}} F = \mathcal{J}_{\mathbf{H}} F$.

Finally (iii) is an easy corollary of (i) and (ii). \square

The main result of this section is the following:

Theorem 14.16. *Suppose $\eta > 0$. Then the pair of $\mathcal{F}_G : C_\eta^\infty(G/K)_{K\text{-finite}} \rightarrow \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)_{K\text{-finite}}$ and $\mathcal{F}_{\mathbf{H}} : C_\eta^\infty(A) \rightarrow \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$ is an isomorphism of radial pairs:*

$$(\mathcal{F}_G, \mathcal{F}_{\mathbf{H}}) : (C_\eta^\infty(G/K)_{K\text{-finite}}, C_\eta^\infty(A)) \longrightarrow (\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)_{K\text{-finite}}, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)).$$

The inverse morphism is $(\mathcal{J}_G, \mathcal{J}_{\mathbf{H}})$. Therefore, if $V \in \widehat{K}_M$ then

(i) for any $\Phi \in \text{Hom}_K(V, C_c^\infty(G/K))$ it holds that

$$\begin{aligned} \int_{G/K} \Phi[v](x) e^{(-i\lambda + \rho)(A(\bar{w}^{-1}x))} dx \\ = \int_A \Phi[v](a) \mathbf{G}(-i\lambda, w^{-1}a) \prod_{\alpha \in \Sigma^+} |2 \sinh \alpha(\log a)|^{\dim \mathfrak{g}_\alpha} da \\ \forall v \in V_{\text{single}}^M, \forall \lambda \in \mathfrak{a}^* \text{ and } \forall w \in W; \end{aligned}$$

(ii) for any $\Phi \in \text{Hom}_K(V, C_c^\infty(G/K))$ such that

$$\Phi[v](a) = 0 \quad \forall v \in V_{\text{double}}^M \text{ and } \forall a \in A$$

it holds that

$$\int_{G/K} \Phi[v](x) e^{(-i\lambda + \rho)(A(\bar{w}^{-1}x))} dx = 0 \quad \forall v \in V_{\text{double}}^M, \forall \lambda \in \mathfrak{a}^* \text{ and } \forall w \in W.$$

Proof. It suffices to show the inverse

$$(\mathcal{J}_G, \mathcal{J}_{\mathbf{H}}) : (\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)_{K\text{-finite}}, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)) \longrightarrow (C_\eta^\infty(G/K)_{K\text{-finite}}, C_\eta^\infty(A))$$

is a morphism in \mathcal{C}_{Ch} . Suppose $V \in \widehat{K}_M$ and let $\Phi \in \text{Hom}_K(V, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M))$. Let $\tilde{\Gamma}_B^V$ be as in the proof of Proposition 14.12. For each $\lambda \in \mathfrak{a}^*$ define $\Phi_\lambda \in \text{Hom}_K(V, B_G(i\lambda))$ as in (14.7). Then for $v \in V_{\text{single}}^M$ and $a \in A$ we have

$$\begin{aligned} (\mathcal{J}_G \circ \Phi)[v](a) &= \int_{\mathfrak{a}^*} \left(\int_K e^{(i\lambda + \rho)(A(k^{-1}a))} \Phi[v](\lambda, k) dk \right) |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= \int_{\mathfrak{a}^*} \left(\int_K e^{(i\lambda + \rho)(A(k^{-1}a))} \Phi_\lambda[v](k) dk \right) |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= \int_{\mathfrak{a}^*} \left(\frac{1}{|W|} \sum_{w \in W} \Phi_\lambda[v](\bar{w}) \mathbf{G}(i\lambda, w^{-1}a) \right) |\mathbf{c}(\lambda)|^{-2} d\lambda \quad (\because \text{Theorem 12.3 (i)}) \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathfrak{a}^*} \left(\frac{1}{|W|} \sum_{w \in W} \tilde{\Gamma}_B^V(\Phi)[v](\lambda, w) \mathbf{G}(i\lambda, w^{-1}a) \right) |\mathbf{c}(\lambda)|^{-2} d\lambda \\
&= (\mathcal{J}_{\mathbf{H}} \circ \tilde{\Gamma}_B^V(\Phi))[v](a).
\end{aligned}$$

Thus $\tilde{\Gamma}_0^V(\mathcal{J}_G \circ \Phi) = \mathcal{J}_{\mathbf{H}} \circ \tilde{\Gamma}_B^V(\Phi)$. Next, suppose $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M))$. Then for $v \in V_{\text{double}}^M$ and $\lambda \in \mathfrak{a}^*$ it holds that

$$\Phi_\lambda[v](1) = \Phi[v](\lambda, 1) = 0.$$

Hence from Theorem 12.3 (ii)' we have for $v \in V_{\text{double}}^M$ and $a \in A$

$$(\mathcal{J}_G \circ \Phi)[v](a) = \int_{\mathfrak{a}^*} \left(\int_K e^{(i\lambda + \rho)(A(k^{-1}a))} \Phi_\lambda[v](k) dk \right) |\mathbf{c}(\lambda)|^{-2} d\lambda = 0.$$

This shows $\mathcal{J}_G \circ \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, C_\eta^\infty(G/K))$. Thus we have proved $(\mathcal{J}_G, \mathcal{J}_{\mathbf{H}})$ satisfies Conditions (Ch-1) and (Ch-2) in Definition 8.1. \square

To relate two Plancherel formulas given in Proposition 14.9 (iii) and Proposition 14.15 (iii), we should mention that the inner product on $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times K/M)_{K\text{-finite}}$ defined by

$$\int_{\mathfrak{a}^*} \int_K F_1(\lambda, k) \overline{F_2(\lambda, k)} dk |\mathbf{c}(\lambda)|^{-2} d\lambda$$

and the inner product on $\widetilde{\text{PW}}_\eta(\mathfrak{a}^* \times W)$ defined by

$$\int_{\mathfrak{a}^*} \frac{1}{|W|} \sum_{w \in W} F_1(\lambda, w) \overline{F_2(\lambda, w)} |\mathbf{c}(\lambda)|^{-2} d\lambda$$

are compatible with restriction in the sense of Definition 9.9. This is indeed immediate from Proposition 9.10.

15. THE CHEVALLEY RESTRICTION THEOREM, II

In this section, we shall generalize Theorem 2.3 to the case where $\mathcal{F} = \mathcal{A}$, the class of analytic functions. We start with generalization of the invariant case.

Theorem 15.1. *The restriction map $\gamma_0 : C^\infty(G/K) \rightarrow C^\infty(A)$ induces the bijection*

$$\gamma_0 : \mathcal{A}(G/K)^K \xrightarrow{\sim} \mathcal{A}(A)^W.$$

As far as the author knows, the theorem had been open. Since this can be considered as a local version of Helgason's result [Hel3, Lemma 2.2], the first half of our proof follows his idea.

Proof of Theorem 15.1. In view of (1.14) and (1.15), γ_0 is identified with the restriction map $C^\infty(\mathfrak{s}) \rightarrow C^\infty(\mathfrak{a})$. Hence we show that the latter interpretation of γ_0 induces the bijection

$$\gamma_0 : \mathcal{A}(\mathfrak{s})^K \xrightarrow{\sim} \mathcal{A}(\mathfrak{a})^W.$$

It is trivial that $\gamma_0(\mathcal{A}(\mathfrak{s})^K) \subset \mathcal{A}(\mathfrak{a})^W$ and we already know any $\varphi \in \mathcal{A}(\mathfrak{a})^W$ is lifted to a unique $f \in C^\infty(\mathfrak{s})^K$ such that $\gamma_0(f) = \varphi$. Hence the only non-trivial point is

the analyticity of f . Because of the K -invariance of f it is enough to show that f is analytic in a neighborhood of each $x \in \mathfrak{a}$ such that $\alpha(x) \geq 0$ for $\alpha \in \Pi$. Put $\Theta = \{\alpha \in \Pi; \alpha(x) = 0\}$, $\mathfrak{a}(\Theta) = \sum_{\alpha \in \Theta} \mathbb{R}\alpha^\vee \subset \mathfrak{a}$ and $\alpha^\Theta = \{H \in \mathfrak{a}; \alpha(H) = 0 \text{ for any } \alpha \in \Theta\}$. Let $W(\Theta)$ be the subgroup of W generated by $\{s_\alpha; \alpha \in \Theta\}$. Let $\ell = |\Pi|$, $k = |\Theta|$ and $\Theta = \{\alpha_1, \dots, \alpha_k\}$. Take linearly independent $\ell - k$ elements $\varpi_{k+1}, \dots, \varpi_\ell \in \mathfrak{a}^*$ so that $\varpi_i|_{\mathfrak{a}(\Theta)} = 0$ for $i = k+1, \dots, \ell$. Since $\{\alpha_1, \dots, \alpha_k, \varpi_{k+1}, \dots, \varpi_\ell\}$ is a coordinate system of \mathfrak{a} , we can expand φ into a power series of the form

$$\sum_{\nu=(\nu_1, \dots, \nu_\ell)} c_\nu \alpha_1^{\nu_1} \cdots \alpha_k^{\nu_k} (\varpi_{k+1} - \varpi_{k+1}(x))^{\nu_{k+1}} \cdots (\varpi_\ell - \varpi_\ell(x))^{\nu_\ell}$$

which converges absolutely and uniformly on a complex open neighborhood $U \subset \mathfrak{a}_\mathbb{C}$ of x and coincides with φ on $U \cap \mathfrak{a}$. Take a set $\{j_1, \dots, j_k\}$ of algebraically independent homogeneous generators of $\mathcal{P}(\mathfrak{a}(\Theta))^{W(\Theta)}$. Then the holomorphic map

$$j : \mathfrak{a}_\mathbb{C} = \mathfrak{a}(\Theta)_\mathbb{C} \times \mathfrak{a}_\mathbb{C}^\Theta \ni (H_1, H_2) \longmapsto ((j_1(H_1), \dots, j_k(H_1)), H_2) \in \mathbb{C}^k \times \mathfrak{a}_\mathbb{C}^\Theta$$

is proper (the inverse image of any compact set is compact) by [Ko1, Lemma 7]. From this fact and the well-known surjectivity of j it follows that the usual topology of $\mathbb{C}^k \times \mathfrak{a}_\mathbb{C}^\Theta$ coincides with the quotient topology by j . Furthermore, since a fiber of j is a $W(\Theta)$ -orbit one easily sees j is an open map. In particular $j(U)$ is open. Now we put for $m = 0, 1, 2, \dots$

$$\varphi_m = \sum_{\nu_1 + \dots + \nu_\ell = m} c_\nu \alpha_1^{\nu_1} \cdots \alpha_k^{\nu_k} (\varpi_{k+1} - \varpi_{k+1}(x))^{\nu_{k+1}} \cdots (\varpi_\ell - \varpi_\ell(x))^{\nu_\ell}.$$

Since each φ_m is a $W(\Theta)$ -invariant polynomial, there exists a polynomial ψ_m on $\mathbb{C}^k \times \mathfrak{a}_\mathbb{C}^\Theta$ such that $\varphi_m = \psi_m \circ j$. Then as a limit of uniform convergence, $\psi := \sum_m \psi_m$ is a holomorphic function on $j(U)$ such that $\varphi = \psi \circ j$.

Let $\mathfrak{g}(\Theta)$ be the Lie subalgebra of \mathfrak{g} generated by \mathfrak{m} and \mathfrak{g}_α for $\alpha \in \Sigma$ with $\alpha(x) = 0$. This is reductive and θ -stable. Put $\mathfrak{k}(\Theta) = \mathfrak{k} \cap \mathfrak{g}(\Theta)$, $\mathfrak{s}(\Theta) = \mathfrak{s} \cap \mathfrak{g}(\Theta)$ and let $K(\Theta) \subset G$ be the analytic subgroup of $\mathfrak{k}(\Theta)$. Note $\mathfrak{s}(\Theta) \cap \mathfrak{a} = \mathfrak{a}(\Theta)$ is a maximal Abelian subspace of $\mathfrak{s}(\Theta)$ and the Weyl group for $(\mathfrak{g}(\Theta), \mathfrak{a}(\Theta))$ is naturally identified with $W(\Theta)$. Hence by the classical Chevalley restriction theorem, for each $i = 1, \dots, k$ there exists $J_i \in \mathcal{P}(\mathfrak{s}(\Theta))^{K(\Theta)}$ such that $J_i|_{\mathfrak{a}(\Theta)} = j_i$. Define the holomorphic map

$$J : \mathfrak{a}_\mathbb{C} = \mathfrak{s}(\Theta)_\mathbb{C} \times \mathfrak{a}_\mathbb{C}^\Theta \ni (X, H_2) \longmapsto ((J_1(X), \dots, J_k(X)), H_2) \in \mathbb{C}^k \times \mathfrak{a}_\mathbb{C}^\Theta.$$

Then $\psi \circ J$ is holomorphic on $J^{-1}(j(U))$ and for any $k \in K(\Theta)$ and $(H_1, H_2) \in U \cap \mathfrak{a}$ it holds that

$$(\psi \circ J)(k(H_1, H_2)) = (\psi \circ J)(kH_1, H_2) = (\psi \circ j)(H_1, H_2) = \varphi(H_1, H_2) = f(k(H_1, H_2)).$$

Since $\mathfrak{s}(\Theta)$ has a $K(\Theta)$ -invariant metric and since each $K(\Theta)$ -orbit in $\mathfrak{s}(\Theta)$ intersects with $\mathfrak{a}(\Theta)$, one has $\mathcal{B} = \text{Ad}(K(\Theta))(\mathcal{B} \cap \mathfrak{a}(\Theta))$ for any open ball $\mathcal{B} \subset \mathfrak{s}(\Theta)$ with center 0. Hence we can take an open neighborhood U' of x in $\mathfrak{s}(\Theta) \times \mathfrak{a}^\Theta$ ($\simeq \mathfrak{s}(\Theta) \oplus \mathfrak{a}^\Theta \subset \mathfrak{s}$) so that $U' \subset \text{Ad}(K(\Theta))(U \cap \mathfrak{a})$. The above calculation shows $f|_{U'}$ is analytic.

Now put

$$\mathfrak{k}^\Theta := \sum_{\alpha \in \Sigma; \alpha(x) > 0} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}, \quad \mathfrak{s}^\Theta := \sum_{\alpha \in \Sigma; \alpha(x) > 0} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{s}$$

and consider the analytic map

$$L : \mathfrak{k}^\Theta \times (\mathfrak{s}(\Theta) \times \mathfrak{a}^\Theta) \ni (Y, X) \longmapsto \text{Ad}(\exp Y)(X) \in \mathfrak{s}.$$

We assert L is a local diffeomorphism at $(0, x)$. In fact, the tangent spaces of both sides are naturally identified with themselves at each point, and $\mathfrak{s} = \mathfrak{s}^\Theta \oplus (\mathfrak{s}(\Theta) \times \mathfrak{a}^\Theta)$. Hence the assertion follows from

$$dL_{(0,x)}(X_\alpha + \theta X_{-\alpha}, 0) = \frac{d}{dt} \text{Ad}(\exp t(X_\alpha + \theta X_{-\alpha}))(x) \Big|_{t=0} = -\alpha(x)(X_\alpha - \theta X_{-\alpha})$$

for $\alpha \in \Sigma$ with $\alpha(x) > 0$ and $X_\alpha \in \mathfrak{g}_\alpha$,

$$dL_{(0,x)}(0, X) = \frac{d}{dt}(x + tX) \Big|_{t=0} = X \quad \text{for } X \in \mathfrak{s}(\Theta) \times \mathfrak{a}^\Theta.$$

Take an open neighborhood U'' of $(0, x) \in \mathfrak{k}^\Theta \times (\mathfrak{s}(\Theta) \times \mathfrak{a}^\Theta)$ so that $U'' \subset \mathfrak{k}^\Theta \times U'$ and U'' is analytically diffeomorphic to $L(U'')$ by L . Since $(f \circ L)(Y, X) = f(X)$ for $(Y, X) \in U''$, $f \circ L$ is analytic on U'' . Hence $f = f \circ L \circ (L|_{U''})^{-1}$ is analytic on the open neighborhood $L(U'')$ of $x \in \mathfrak{s}$. \square

Let (σ, U) be a finite-dimensional representation of W and $\{u_1, \dots, u_m\}$ a basis of U ($m = \dim U$). Let $H_W(\mathfrak{a}_\mathbb{C}^*) \subset \mathcal{P}(\mathfrak{a})$ be the space of W -harmonic polynomials on \mathfrak{a} . It is well known that $\dim \text{Hom}_W(U, H_W(\mathfrak{a}_\mathbb{C}^*)) = m$. Let $\{\varphi_1, \dots, \varphi_m\}$ be a basis of $\text{Hom}_W(U, H_W(\mathfrak{a}_\mathbb{C}^*))$.

Lemma 15.2. *Put $\mathbf{k}_\sigma(\alpha) = (m - \text{Trace } \sigma(s_\alpha))/2$ for each $\alpha \in R_1$. Then there exists a non-zero constant C such that*

$$\det(\varphi_j[u_i])_{1 \leq i, j \leq m} = C \prod_{\alpha \in R_1^+} \alpha^{\mathbf{k}_\sigma(\alpha)}.$$

Proof. The determinant is non-zero by [HC, §2] (cf. [V, Ch. 4, Exercise 70 (d)] and [M, Lemma 5.2.1]) and has degree $\sum_{\alpha \in R_1^+} \mathbf{k}_\sigma(\alpha)$ by [Br, Formula (1)]. Hence it suffices to prove the determinant is divided by $\alpha^{\mathbf{k}_\sigma(\alpha)}$ for each $\alpha \in R_1^+$. If $U^{-s_\alpha} \subset U$ denotes the -1 -eigenspace of $\sigma(s_\alpha)$ then $\dim U^{-s_\alpha} = \mathbf{k}_\sigma(\alpha)$. We may assume $\{u_1, \dots, u_{\mathbf{k}_\sigma(\alpha)}\}$ is a basis of U^{-s_α} . Since $\varphi_j[u_i]$ is divided by α for each $i = 1, \dots, \mathbf{k}_\sigma(\alpha)$ and $j = 1, \dots, m$, $\det(\varphi_j[u_i])_{1 \leq i, j \leq m}$ is divided by $\alpha^{\mathbf{k}_\sigma(\alpha)}$. \square

Lemma 15.3. *For any $\varphi \in \text{Hom}_W(U, \mathcal{A}(\mathfrak{a}))$ there exist $c_1, \dots, c_m \in \mathcal{A}(\mathfrak{a})^W$ such that*

$$(15.1) \quad \varphi = c_1 \varphi_1 + \dots + c_m \varphi_m.$$

Proof. Let $Q = (q_{ij}) \in \text{Mat}(m, m; \mathcal{P}(\mathfrak{a}))$ be the cofactor matrix of $(\varphi_j[u_i])_{1 \leq i, j \leq m}$ and denote the determinant of Lemma 15.2 by D . In general if $\varphi \in \text{Hom}_W(U, \mathcal{A}(\mathfrak{a}))$ and $c_1, \dots, c_m \in \mathcal{A}(\mathfrak{a})$ satisfy (15.1) then

$$(15.2) \quad c_i = \frac{1}{D} \sum_{j=1}^m q_{ij} \varphi_j[u_j] \quad \text{for } i = 1, \dots, m$$

and hence

$$(15.3) \quad \sum_{j=1}^m q_{ij} \varphi[u_j] \text{ is divided by } D \text{ for } i = 1, \dots, m.$$

Conversely if $\varphi \in \text{Hom}_W(U, \mathcal{A}(\mathfrak{a}))$ satisfies (15.3) then (15.1) holds for $\{c_i\}$ defined by (15.2). (In this case $\{c_i\} \subset \mathcal{A}(\mathfrak{a})^W$ since for any $w \in W$, $c'_i := wc_i$ ($i = 1, \dots, m$) also satisfy (15.1) and hence (15.2).) By Lemma 15.2, (15.3) is still equivalent to

$$(15.4) \quad \partial(\alpha^\vee)^k \left(\sum_{j=1}^m q_{ij} \varphi[u_j] \right) \Big|_{\alpha=0} = 0 \quad \text{for } \begin{cases} \alpha \in R_1^+, k = 0, \dots, \mathbf{k}_\sigma(\alpha) - 1, \\ \text{and } i = 1, \dots, m. \end{cases}$$

Now for any $\varphi \in \text{Hom}_W(U, \mathcal{A}(\mathfrak{a}))$ and $u \in U$ expand $\varphi[u]$ into the Taylor series at 0 and let $\varphi^{(d)}[u]$ be the homogeneous part of the series with degree d ($d = 0, 1, 2, \dots$). Thus there is a complex neighborhood of $0 \in \mathfrak{a}_\mathbb{C}$ on which $\sum_{d=0}^\infty \varphi^{(d)}[u]$ converges absolutely and uniformly for all $u \in U$. Clearly the map $\varphi^{(d)} : U \ni u \mapsto \varphi^{(d)}[u] \in \mathcal{P}(\mathfrak{a})$ is a W -homomorphism. Since $\text{Hom}_W(U, \mathcal{P}(\mathfrak{a})) = \bigoplus_{j=1}^m \mathcal{P}(\mathfrak{a})^W \varphi_j$, each $\varphi^{(d)}$ satisfies (15.4). Taking the sum over d we conclude φ also satisfies (15.4) around 0, namely for $\alpha \in R_1^+$, $k = 0, \dots, \mathbf{k}_\sigma(\alpha) - 1$, and $i = 1, \dots, m$ there exists an open neighborhood Y of 0 in the hyperplane $\alpha = 0$ such that

$$\partial(\alpha^\vee)^k \left(\sum_{j=1}^m q_{ij} \varphi[u_j] \right) \Big|_Y = 0.$$

This implies (15.4) for φ since the left-hand side of (15.4) is analytic on $\alpha = 0$. \square

Theorem 15.4. *Theorem 2.3 is valid for $\mathcal{F} = \mathcal{A}$.*

Proof. All except the second assertion of (iv) are trivial. But the “only if” part of the assertion reduces to the polynomial case (see the last part of the proof of [O, Theorem 3.5]). Hence we have only to show for any $V \in \widehat{K}_M$ and $\varphi \in \text{Hom}_W(V^M, \mathcal{A}(A)) \simeq \text{Hom}_W(V^M, \mathcal{A}(\mathfrak{a}))$ with $\varphi[V_{\text{double}}^M] = \{0\}$ there exists $\Phi \in \text{Hom}_W(V^M, \mathcal{A}(\mathfrak{s}))$ such that $\Gamma_0^V(\Phi) = \varphi$. Take a basis $\{\varphi_1, \dots, \varphi_m\}$ of $\text{Hom}_W(V_{\text{single}}^M, \mathcal{P}(\mathfrak{a}))$ and extend each φ_i to an element of $\text{Hom}_W(V^M, \mathcal{P}(\mathfrak{a}))$ by letting $\varphi[V_{\text{double}}^M] = \{0\}$. Applying Lemma 15.3 to the case of $U = V_{\text{single}}^M$, we obtain $c_1, \dots, c_m \in \mathcal{A}(\mathfrak{a})^W$ such that $\varphi = c_1\varphi_1 + \dots + c_m\varphi_m$. Now for each $i = 1, \dots, m$ we have $C_i \in \mathcal{A}(\mathfrak{s})^K$ such that $\gamma_0(C_i) = c_i$ by Theorem 15.1 and $\Phi_i \in \text{Hom}_K(V, \mathcal{P}(\mathfrak{s}))$ such that $\Gamma_0^V(\Phi_i) = \varphi_i$ by Theorem 2.3 for $\mathcal{F} = \mathcal{P}$. Hence $\Phi := \sum_i C_i \Phi_i \in \text{Hom}_K(V, \mathcal{A}(\mathfrak{s}))$ satisfies $\Gamma_0^V(\Phi) = \varphi$. \square

Corollary 15.5. *$(\mathcal{A}(G/K), \mathcal{A}(A))$ is a radial pair with radial restriction γ_0 .*

Proof. By the theorem the pair is an object of \mathcal{C}_{Ch} and γ_0 is its radial restriction. Furthermore the pair is a subobject of $(C^\infty(G/K), C^\infty(A)) \in \mathcal{C}_{\text{rad}}$. Hence it belongs to \mathcal{C}_{rad} by Proposition 8.12 (ii). \square

In particular the correspondence

$$\Xi_0^{\min} : \{\mathbf{H}\text{-submodules of } C^\infty(A)\} \rightarrow \{(\mathfrak{g}_\mathbb{C}, K)\text{-submodules of } C^\infty(G/K)_{K\text{-finite}}\}$$

defined in Definition 8.21 restricts to

$$(15.5) \quad \Xi_0^{\min} : \{\mathbf{H}\text{-submodules of } \mathcal{A}(A)\} \rightarrow \{(\mathfrak{g}_{\mathbb{C}}, K)\text{-submodules of } \mathcal{A}(G/K)_{K\text{-finite}}\}.$$

In §18 the latter correspondence will be extended to a functor Ξ sending any \mathbf{H} -module to a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module.

16. THE FUNCTOR Ξ_{rad}

In Definition 8.16 we introduced the functor $\Xi_{\text{w-rad}}$ which sends an \mathbf{H} -module \mathcal{X} to a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module

$$\Xi_{\text{w-rad}}(\mathcal{X}) = \bigoplus_{V \in \widehat{K}_M} \bar{P}_G(V) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X}).$$

The pair $(\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X})$ is a weak radial pair with the universal property stated in Proposition 8.20. In this section we study a functor Ξ_{rad} which has similar properties for the category \mathcal{C}_{rad} of radial pairs.

Definition 16.1 (the functor Ξ_{rad}). Suppose $\mathcal{X} \in \mathbf{H}\text{-Mod}$. Let $\mathcal{N}_{\text{rad}}(\mathcal{X})$ be the \mathbb{C} -linear subspace of $\Xi_{\text{w-rad}}(\mathcal{X})$ spanned by

$$(16.1) \quad \bar{\Psi}(D) \otimes \varphi - D \otimes (\varphi \circ \tilde{\Gamma}(\Psi)) \quad \text{with} \quad \begin{cases} F, V \in \widehat{K}_M, \Psi \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(V), P_G(F)), \\ D \in \bar{P}_G(V), \varphi \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(F_{\text{single}}^M), \mathcal{X}), \end{cases}$$

where $\bar{\Psi}$ is the image of Ψ under (8.6). Note $\mathcal{N}_{\text{rad}}(\mathcal{X})$ is stable under the $(\mathfrak{g}_{\mathbb{C}}, K)$ -action. We put

$$\Xi_{\text{rad}}(\mathcal{X}) = \Xi_{\text{w-rad}}(\mathcal{X}) / \mathcal{N}_{\text{rad}}(\mathcal{X}) \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}.$$

We give $(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})$ a structure of a weak radial pair first. Recall the linear maps $\tilde{\gamma}^{\bar{P}_G(V)} : \bar{P}_G(V) \rightarrow P_{\mathbf{H}}(V_{\text{single}}^M)$ ($V \in \widehat{K}_M$) and $\gamma_{\text{w-rad}} : \Xi_{\text{w-rad}}(\mathcal{X}) \rightarrow \mathcal{X}$ defined in Lemma 8.17 (iv) and before Lemma 8.18 respectively. Suppose an element in $\mathcal{N}_{\text{rad}}(\mathcal{X})$ is given by (16.1). Since $\Psi \in \text{Hom}^{2 \rightarrow 2}$, it follows from the commutativity of (6.4) that $\tilde{\gamma}^{\bar{P}_G(F)} \circ \bar{\Psi} = \tilde{\Gamma}(\Psi) \circ \tilde{\gamma}^{\bar{P}_G(V)}$. Hence we have

$$\gamma_{\text{w-rad}}(\bar{\Psi}(D) \otimes \varphi - D \otimes (\varphi \circ \tilde{\Gamma}(\Psi))) = \varphi(\tilde{\gamma}^{\bar{P}_G(F)}(\bar{\Psi}(D))) - (\varphi \circ \tilde{\Gamma}(\Psi))(\tilde{\gamma}^{P_G(V)}(D)) = 0.$$

Combining this with Lemma 8.18, we can apply Proposition 8.26 (ii) to the case where $(\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X}, \gamma_{\text{w-rad}}, \mathcal{N}_{\text{rad}}(\mathcal{X}), \{0\})$ is $(\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}, \gamma_{\mathcal{M}}, \mathcal{Y}, \mathcal{X})$ in the proposition. Hence $(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) = (\Xi_{\text{w-rad}}(\mathcal{X}) / \mathcal{N}_{\text{rad}}(\mathcal{X}), \mathcal{X} / \{0\}) \in \mathcal{C}_{\text{w-rad}}$ and the induced linear map $\gamma_{\text{rad}} : \Xi_{\text{rad}}(\mathcal{X}) \rightarrow \mathcal{X}$ satisfies Conditions (rest-1'), (rest-2) and (rest-3). In particular, the structure of $(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})$ as a weak radial pair is given by the linear map

$$\begin{aligned} \tilde{\Gamma}_{\text{rad}}^V : \text{Hom}_K(V, \Xi_{\text{rad}}(\mathcal{X})) &\longrightarrow \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}); \\ \Phi &\longmapsto (V_{\text{single}}^M \hookrightarrow V \xrightarrow{\Phi} \Xi_{\text{rad}}(\mathcal{X}) \xrightarrow{\gamma_{\text{rad}}} \mathcal{X}) \end{aligned}$$

and the natural surjective map

$$(16.2) \quad \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{w-rad}}(\mathcal{X})) \twoheadrightarrow \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{rad}}(\mathcal{X}))$$

(actually this is a bijective map) defined for each $V \in \widehat{K}_M$.

Theorem 16.2. $(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$.

Proof. By Lemma 8.13 $(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})$ satisfies (rad-1). Let us check (rad-2). Suppose $F, V \in \widehat{K}_M$. Recall

$$\begin{aligned} \text{Hom}_K(V, \Xi_{\text{w-rad}}(\mathcal{X})) &\simeq \bigoplus_{F \in \widehat{K}_M} \text{Hom}_K(V, \bar{P}_G(F)) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(F_{\text{single}}^M), \mathcal{X}), \\ \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{w-rad}}(\mathcal{X})) &= I_V \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X}), \end{aligned}$$

where $I_V \in \text{Hom}_K(V, \bar{P}_G(V))$ denotes the map $V \ni v \mapsto 1 \otimes v \in \bar{P}_G(V)$. By (16.2) any $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{rad}}(\mathcal{X}))$ is written as $I_V \otimes \varphi$ with $\varphi \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X})$ (we omit “mod $\mathcal{N}_{\text{rad}}(\mathcal{X})$ ”). Now for any $\Psi \in \text{Hom}_K^{2 \rightarrow 2}(E, P_G(V))$ and $e \in E$

$$(\Phi \circ \Psi)[e] = \bar{\Psi}[e] \otimes \varphi = (1 \otimes e) \otimes (\varphi \circ \tilde{\Gamma}(\Psi))$$

by (16.1). Thus $\Phi \circ \Psi = I_E \otimes (\varphi \circ \tilde{\Gamma}(\Psi)) \in \text{Hom}_K^{2 \rightarrow 2}(E, \Xi_{\text{rad}}(\mathcal{X}))$. This proves (rad-2). \square

Lemma 16.3. Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{rad}}$ and $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : (\Xi_{\text{w-rad}}(\mathcal{X}), \mathcal{X}) \rightarrow \mathcal{M}$ is a morphism of $\mathcal{C}_{\text{w-rad}}$. Then $\mathcal{N}_{\text{rad}}(\mathcal{X}) \subset \text{Ker } \mathcal{I}_G$ and hence the morphism $(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) \rightarrow \mathcal{M}$ of \mathcal{C}_{rad} is naturally induced.

Proof. For $F, V \in \widehat{K}_M$, $\Psi \in \text{Hom}_{\mathbf{g}_{\mathbf{C}}, K}^{2 \rightarrow 2}(P_G(V), P_G(F))$ and $\varphi \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(F_{\text{single}}^M), \mathcal{X})$ define $\Phi \in \text{Hom}_K(V, \mathcal{N}_{\text{rad}}(\mathcal{X}))$ by

$$V \ni v \longmapsto \bar{\Psi}[v] \otimes \varphi - (1 \otimes v) \otimes (\varphi \circ \tilde{\Gamma}(\Psi)) \in \mathcal{N}_{\text{rad}}(\mathcal{X}).$$

We assert $\mathcal{I}_G \circ \Phi = 0$. Since $\Phi = \bar{\Psi} \otimes \varphi - I_V \otimes (\varphi \circ \tilde{\Gamma}(\Psi))$ we have

$$\begin{aligned} \tilde{\Gamma}_{\mathcal{M}}^V(\mathcal{I}_G \circ \Phi) &= \mathcal{I}_{\mathbf{H}} \circ \tilde{\Gamma}_{\text{w-rad}}^V(\Phi) && (\because (\text{Ch-1}) \text{ for } \mathcal{I}_G) \\ &= \mathcal{I}_{\mathbf{H}} \circ (\varphi \circ \tilde{\Gamma}(\Psi) - \varphi \circ \tilde{\Gamma}(\Psi)) && (\because (8.8)) \\ &= 0. \end{aligned}$$

Since $I_F \otimes \varphi \in \text{Hom}_K^{2 \rightarrow 2}(F, \Xi_{\text{w-rad}}(\mathcal{X}))$ and $I_V \otimes (\varphi \circ \tilde{\Gamma}(\Psi)) \in \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{w-rad}}(\mathcal{X}))$ we also have

$$\begin{aligned} \mathcal{I}_G \circ (I_F \otimes \varphi) &\in \text{Hom}_K^{2 \rightarrow 2}(F, \mathcal{M}_G), && (\because (\text{Ch-2}) \text{ for } \mathcal{I}_G) \\ \mathcal{I}_G \circ (I_F \otimes \varphi) \circ \Psi &\in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G), && (\because (\text{rad-2}) \text{ for } \mathcal{M}_G) \\ \mathcal{I}_G \circ (I_V \otimes (\varphi \circ \tilde{\Gamma}(\Psi))) &\in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G), && (\because (\text{Ch-2}) \text{ for } \mathcal{I}_G) \\ \therefore \mathcal{I}_G \circ \Phi &= \mathcal{I}_G \circ (I_F \otimes \varphi) \circ \Psi - \mathcal{I}_G \circ (I_V \otimes (\varphi \circ \tilde{\Gamma}(\Psi))) \\ &\in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G). \end{aligned}$$

These facts imply $\mathcal{I}_G \circ \Phi = 0$ since $\tilde{\Gamma}_{\mathcal{M}}^V$ is injective on $\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$. Thus we get our assertion, showing \mathcal{I}_G maps any element given by (16.1) to zero. \square

From the lemma we can see in particular that the correspondence $\mathbf{H}\text{-Mod} \ni \mathcal{X} \rightarrow (\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$ is a functor. The following is an easy corollary of Proposition 8.20 and Lemma 16.3.

Theorem 16.4. *The functor $\mathbf{H}\text{-Mod} \ni \mathcal{X} \mapsto (\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$ is left adjoint to the functor $\mathcal{C}_{\text{rad}} \ni (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \mapsto \mathcal{M}_{\mathbf{H}} \in \mathbf{H}\text{-Mod}$. More precisely, if $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{rad}}$ and an \mathbf{H} -homomorphism $\mathcal{I}_{\mathbf{H}} : \mathcal{X} \rightarrow \mathcal{M}_{\mathbf{H}}$ are given, then there exists a unique $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I}_G : \Xi_{\text{rad}}(\mathcal{X}) \rightarrow \mathcal{M}_G$ such that $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : (\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) \rightarrow (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ is a morphism of \mathcal{C}_{rad} .*

Definition 16.5. Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We say $\mathcal{X} \in \mathbf{H}\text{-Mod}$ has a central character $[\lambda]$ if

$$(\Delta - \Delta(\lambda))x = 0 \quad \text{for any } \Delta \in S(\mathfrak{a}_{\mathbb{C}})^W \text{ and } x \in \mathcal{X}.$$

We say $\mathcal{X} \in \mathbf{H}\text{-Mod}$ has a generalized central character $[\lambda]$ if for any $x \in \mathcal{X}$ and $\Delta \in S(\mathfrak{a}_{\mathbb{C}})^W$ there exists a positive integer n such that

$$(\Delta - \Delta(\lambda))^n x = 0.$$

Fix a maximal Abelian subalgebra \mathfrak{b} of \mathfrak{m} . Then $\mathfrak{h} = \mathfrak{b} + \mathfrak{a}$ is a Cartan subalgebra of \mathfrak{g} . Let $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the Weyl group for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $\Upsilon : U(\mathfrak{g}_{\mathbb{C}})^G \xrightarrow{\sim} S(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})}$ the Harish-Chandra isomorphism for the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Suppose $\mu \in \mathfrak{h}_{\mathbb{C}}^*$. We say $\mathcal{Y} \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ has an infinitesimal character $[\mu]$ if

$$(\Delta - \Upsilon(\Delta)(\mu))y = 0 \quad \text{for any } \Delta \in U(\mathfrak{g}_{\mathbb{C}})^G \text{ and } y \in \mathcal{Y}.$$

We say $\mathcal{Y} \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ has a generalized infinitesimal character $[\mu]$ if for any $y \in \mathcal{Y}$ and $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^G$ there exists a positive integer n such that

$$(\Delta - \Upsilon(\Delta)(\mu))^n y = 0.$$

Theorem 16.6. *Fix a positive system of the root system for $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ and let $\rho_{\mathfrak{m}} \in \mathfrak{b}_{\mathbb{C}}^*$ be half the sum of positive roots. Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Let $(\rho_{\mathfrak{m}}, \lambda)$ denote the element of $\mathfrak{h}_{\mathbb{C}}^*$ which equals $\rho_{\mathfrak{m}}$ on \mathfrak{b} and λ on \mathfrak{a} . If $\mathcal{X} \in \mathbf{H}\text{-Mod}$ has (resp., generalized) central character $[\lambda]$ then $\Xi_{\text{rad}}(\mathcal{X})$ has (resp., generalized) infinitesimal character $[(\rho_{\mathfrak{m}}, \lambda)]$.*

Proof. First we assert

$$(16.3) \quad \Upsilon(\Delta)((\rho_{\mathfrak{m}}, \lambda)) = \gamma(\Delta)(\lambda) \quad \text{for any } \Delta \in U(\mathfrak{g}_{\mathbb{C}})^G.$$

Since $U(\mathfrak{g}_{\mathbb{C}})^{MA} \subset \mathfrak{n}_{\mathbb{C}}U(\mathfrak{g}_{\mathbb{C}}) \oplus U(\mathfrak{m}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})^{MA}$, the projection to the second summand defines a linear map (actually an algebra homomorphism)

$$\Upsilon' : U(\mathfrak{g}_{\mathbb{C}})^{MA} \rightarrow U(\mathfrak{m}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})^{MA} \simeq U(\mathfrak{m}_{\mathbb{C}})^M \otimes S(\mathfrak{a}_{\mathbb{C}}).$$

If we define two more algebra homomorphisms

$$\Upsilon'' : U(\mathfrak{m}_{\mathbb{C}})^M \otimes S(\mathfrak{a}_{\mathbb{C}}) \xrightarrow{\text{H.-C.isom.for } \mathfrak{m}_{\mathbb{C}} \otimes \text{shift by } -\rho} S(\mathfrak{b}_{\mathbb{C}})^{W(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})} \otimes S(\mathfrak{a}_{\mathbb{C}}) \hookrightarrow S(\mathfrak{h}_{\mathbb{C}}),$$

$$\begin{aligned} \Upsilon''' : U(\mathfrak{m}_{\mathbb{C}})^M \otimes S(\mathfrak{a}_{\mathbb{C}}) &= ((U(\mathfrak{m}_{\mathbb{C}})\mathfrak{m}_{\mathbb{C}})^M \otimes S(\mathfrak{a}_{\mathbb{C}})) \oplus S(\mathfrak{a}_{\mathbb{C}}) \\ &\xrightarrow{\text{projection to the 2nd summand \& shift by } -\rho} S(\mathfrak{a}_{\mathbb{C}}), \end{aligned}$$

then one easily sees $\Upsilon = \Upsilon'' \circ \Upsilon'|_{U(\mathfrak{g}_{\mathbb{C}})^G}$ and $\gamma|_{U(\mathfrak{g}_{\mathbb{C}})^{MA}} = \Upsilon''' \circ \Upsilon'$. Since the maximal ideal $(U(\mathfrak{m}_{\mathbb{C}})\mathfrak{m}_{\mathbb{C}})^M$ of $U(\mathfrak{m}_{\mathbb{C}})^M$ corresponds to $[\rho_{\mathfrak{m}}]$ by the Harish-Chandra isomorphism for $U(\mathfrak{m}_{\mathbb{C}})^M$, it holds that $\Upsilon''(D)((\rho_{\mathfrak{m}}, \lambda)) = \Upsilon'''(D)(\lambda)$ for any $D \in U(\mathfrak{m}_{\mathbb{C}})^M \otimes S(\mathfrak{a}_{\mathbb{C}})$. Hence we have (16.3).

Secondly let $V \in \widehat{K}_{\text{qsp}}$, $D \in \bar{P}_G(V)$ and $\varphi \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X})$. Then $\Xi_{\text{rad}}(\mathcal{X})$ is spanned by elements like $D \otimes \varphi$. For any $\Delta \in U(\mathfrak{g}_{\mathbb{C}})^G$ define $\Psi_{\Delta} \in \text{End}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(V)) \simeq \text{Hom}_K(V, P_G(V))$ by $\Psi_{\Delta}[v] = \Delta \otimes v$ for $v \in V$. Then $\tilde{\Gamma}(\Psi_{\Delta}) \in \text{End}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M))$ equals the multiplication by $(\Upsilon''' \circ \Upsilon')(\Delta) = \gamma(\Delta) \in S(\mathfrak{a}_{\mathbb{C}})^W$. Thus we have from (16.1)

$$\Delta D \otimes \varphi = \bar{\Psi}_{\Delta}(D) \otimes \varphi = D \otimes (\varphi \circ \tilde{\Gamma}(\Psi_{\Delta})) = D \otimes (\gamma(\Delta)\varphi).$$

Therefore if there exists a positive integer n such that

$$(\gamma(\Delta) - \gamma(\Delta)(\lambda))^n \varphi[v] = 0 \quad \text{for any } v \in V_{\text{single}}^M$$

then

$$\begin{aligned} (\Delta - \Upsilon(\Delta)((\rho_{\mathfrak{m}}, \lambda)))^n D \otimes \varphi &= (\Delta - \gamma(\Delta)(\lambda))^n D \otimes \varphi \\ &= D \otimes ((\gamma(\Delta) - \gamma(\Delta)(\lambda))^n \varphi) = 0. \end{aligned} \quad \square$$

Theorem 16.7. *If $\mathcal{X} \in \mathbf{H}\text{-Mod}$ has finite dimension then $\Xi_{\text{rad}}(\mathcal{X}) \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ has finite length.*

Proof. It suffices to show that $\Xi_{\text{rad}}(\mathcal{X})$ is finitely generated and locally $U(\mathfrak{g}_{\mathbb{C}})^G$ -finite (see [Wa, Proposition 3.7.1 and Theorem 4.2.6]). Since

$$\begin{aligned} \Xi_{\text{rad}}(\mathcal{X}) &= \sum_{V \in \widehat{K}_{\text{qsp}}} \bar{P}_G(V) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V^M)_{\text{single}}, \mathcal{X}) \\ &= U(\mathfrak{g}_{\mathbb{C}}) \sum_{V \in \widehat{K}_{\text{qsp}}} (1 \otimes V) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V^M)_{\text{single}}, \mathcal{X}) \end{aligned}$$

and since there are only finitely many quasi-single-petaled K -types [O, Proposition 3.19], $\sum_{V \in \widehat{K}_{\text{qsp}}} (1 \otimes V) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V^M)_{\text{single}}, \mathcal{X})$ is a finite-dimensional subspace generating $\Xi_{\text{rad}}(\mathcal{X})$. The local finiteness is immediate from Theorem 16.6. \square

The functor Ξ_{rad} can be constructed in a more conceptual way. Put

$$\mathbf{P}_G = \bigoplus_{V \in \widehat{K}_M} P_G(V), \quad \bar{\mathbf{P}}_G = \bigoplus_{V \in \widehat{K}_M} \bar{P}_G(V), \quad \mathbf{P}_{\mathbf{H}} = \bigoplus_{V \in \widehat{K}_M} P_{\mathbf{H}}(V_{\text{single}}^M) = \bigoplus_{V \in \widehat{K}_{\text{qsp}}} P_{\mathbf{H}}(V_{\text{single}}^M).$$

The algebra

$$\text{End}_{\mathfrak{g}_{\mathbb{C}}, K}(\mathbf{P}_G) = \prod_{E \in \widehat{K}_M} \bigoplus_{V \in \widehat{K}_M} \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}(P_G(E), P_G(V))$$

contains

$$\mathbf{A}^{2 \rightarrow 2} := \prod_{E \in \widehat{K}_M} \bigoplus_{V \in \widehat{K}_M} \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(E), P_G(V))$$

as a subalgebra. By Lemma 8.17 (iii), $\bar{\cdot}$ induces the algebra homomorphism

$$\bar{\cdot} : \mathbf{A}^{2 \rightarrow 2} \longrightarrow \text{End}_{\mathfrak{g}_{\mathbb{C}}, K}(\bar{\mathbf{P}}_G).$$

By Proposition 6.3, $\tilde{\Gamma}$ induces the algebra homomorphism

$$\tilde{\Gamma} : \mathbf{A}^{2 \rightarrow 2} \longrightarrow \text{End}_{\mathbf{H}}(\mathbf{P}_{\mathbf{H}}) = \bigoplus_{E, V \in \hat{K}_{\text{qsp}}} \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(E_{\text{single}}^M), P_{\mathbf{H}}(V_{\text{single}}^M)).$$

Let $\mathbf{A}_{\text{op}}^{2 \rightarrow 2}$ denote the opposite algebra of $\mathbf{A}^{2 \rightarrow 2}$. Then $\bar{\mathbf{P}}_G$ is a right $\mathbf{A}_{\text{op}}^{2 \rightarrow 2}$ -module via \cdot and for any $\mathcal{X} \in \mathbf{H}\text{-Mod}$, $\text{Hom}_{\mathbf{H}}(\mathbf{P}_{\mathbf{H}}, \mathcal{X})$ is naturally a left $\mathbf{A}_{\text{op}}^{2 \rightarrow 2}$ -module via $\tilde{\Gamma}$.

Proposition 16.8. *For any $\mathcal{X} \in \mathbf{H}\text{-Mod}$,*

$$(16.4) \quad \Xi_{\text{rad}}(\mathcal{X}) = \bar{\mathbf{P}}_G \otimes_{\mathbf{A}_{\text{op}}^{2 \rightarrow 2}} \text{Hom}_{\mathbf{H}}(\mathbf{P}_{\mathbf{H}}, \mathcal{X}).$$

In particular, Ξ_{rad} is right exact thanks to the projectivity of $\mathbf{P}_{\mathbf{H}}$ in $\mathbf{H}\text{-Mod}$.

Proof. We denote the right-hand side of (16.4) by \mathcal{Y} . First we note

$$\mathcal{Y} = \sum_{E \in \hat{K}_M, V \in \hat{K}_{\text{qsp}}} \bar{P}_G(E) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X}).$$

If $D \in \bar{P}_G(E)$ and $\varphi \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X})$ then

$$D \otimes \varphi = D \otimes (\varphi \circ \tilde{\Gamma}(I_V)) = \bar{I}_V(D) \otimes \varphi = 0$$

unless $E = V$. Here $I_V \in \text{End}_{\mathfrak{g}_{\mathbf{C}}, K}(P_G(V)) \subset \mathbf{A}^{2 \rightarrow 2}$ denotes the identity. Thus

$$\mathcal{Y} = \sum_{V \in \hat{K}_{\text{qsp}}} \bar{P}_G(V) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X})$$

and by this expression a surjective $(\mathfrak{g}_{\mathbf{C}}, K)$ -homomorphism $\pi : \Xi_{\text{w-rad}}(\mathcal{X}) \rightarrow \mathcal{Y}$ is naturally defined. But it is easy to observe that $\text{Ker } \pi = \mathcal{N}_{\text{rad}}(\mathcal{X})$. \square

17. THE FUNCTOR Ξ^{\min}

In this section, we shall define a functor $\Xi^{\min} : \mathbf{H}\text{-Mod} \rightarrow (\mathfrak{g}_{\mathbf{C}}, K)\text{-Mod}$ so that $(\Xi^{\min}(\mathcal{X}), \mathcal{X})$ becomes a radial pair with some canonical radial restriction γ^{\min} . This functor will turn out to extend the correspondence

$$\Xi^{\min} : \{\mathbf{H}\text{-submodules of } P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})\} \rightarrow \{(\mathfrak{g}_{\mathbf{C}}, K)\text{-submodules of } P_G(\mathbb{C}_{\text{triv}})\}$$

defined in Definition 8.21. It will also be shown that when restricted to the category $\mathbf{H}\text{-Mod}^{\text{fd}}$ of the finite-dimensional \mathbf{H} -modules, Ξ^{\min} lifts any sesquilinear pairing.

Suppose $\mathcal{X} \in \mathbf{H}\text{-Mod}$. Recall the linear map $\gamma_{\text{rad}} : \Xi_{\text{rad}}(\mathcal{X}) \rightarrow \mathcal{X}$ used in the beginning of the last section, which satisfies Conditions (rest-1'), (rest-2) and (rest-3) for $(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$. Hence we can apply the correspondence $\Xi_{(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})}^{\natural}$ of Proposition 8.26 to the \mathbf{H} -submodule $\{0\} \subset \mathcal{X}$. Put $\mathcal{N}^{\min}(\mathcal{X}) = \Xi_{(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})}^{\natural}(\{0\})$, namely

$$\mathcal{N}^{\min}(\mathcal{X}) = \sum \{ \mathcal{V} \subset \Xi_{\text{rad}}(\mathcal{X}); \text{ a } K\text{-stable } \mathbb{C}\text{-subspace with } \gamma_{\text{rad}}(\mathcal{V}) = \{0\} \}.$$

Then from Proposition 8.26 (iii) we have

Theorem 17.1 (the functor Ξ^{\min}). *We put*

$$\Xi^{\min}(\mathcal{X}) = \Xi_{\text{rad}}(\mathcal{X}) / \mathcal{N}^{\min}(\mathcal{X}) \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}.$$

Then $(\Xi^{\min}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$. A linear map $\gamma^{\min} : \Xi^{\min}(\mathcal{X}) \rightarrow \mathcal{X}$ naturally induced from γ_{rad} is a radial restriction of $(\Xi^{\min}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$ in the sense of Definition 8.7. Furthermore γ^{\min} satisfies Condition (rest-3) in Lemma 8.13.

Lemma 17.2. *Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{rad}}$ has a radial restriction $\gamma_{\mathcal{M}}$ satisfying (rest-3). Suppose moreover $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : (\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X}) \rightarrow \mathcal{M}$ is a morphism of \mathcal{C}_{rad} . Then it holds that*

$$(17.1) \quad \text{Ker } \mathcal{I}_G = \Xi_{(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})}^{\natural}(\text{Ker } \mathcal{I}_{\mathbf{H}}).$$

In particular, $\mathcal{N}^{\min}(\mathcal{X}) \subset \text{Ker } \mathcal{I}_G$ and hence a morphism $(\Xi^{\min}(\mathcal{X}), \mathcal{X}) \rightarrow \mathcal{M}$ of \mathcal{C}_{rad} is naturally induced.

Proof. We first prove

$$(17.2) \quad \gamma_{\mathcal{M}} \circ \mathcal{I}_G = \mathcal{I}_{\mathbf{H}} \circ \gamma_{\text{rad}}.$$

For $V \in \widehat{K}_M$ and $\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{rad}}(\mathcal{X}))$ we have $\tilde{\Gamma}_{\mathcal{M}}^V(\mathcal{I}_G \circ \Phi) = \mathcal{I}_{\mathbf{H}} \circ \tilde{\Gamma}_{\text{rad}}^V(\Phi)$ by (Ch-1). Hence $(\gamma_{\mathcal{M}} \circ \mathcal{I}_G \circ \Phi)|_{V_{\text{single}}^M} = (\mathcal{I}_{\mathbf{H}} \circ \gamma_{\text{rad}} \circ \Phi)|_{V_{\text{single}}^M}$. Since $\mathcal{I}_G \circ \Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G)$ by (Ch-2), Condition (rest-1) for $\gamma_{\mathcal{M}}$ and Condition (rest-1') for γ_{rad} imply $(\gamma_{\mathcal{M}} \circ \mathcal{I}_G \circ \Phi)|_{V_{\text{double}}^M} = (\mathcal{I}_{\mathbf{H}} \circ \gamma_{\text{rad}} \circ \Phi)|_{V_{\text{double}}^M} = 0$. Thus $(\gamma_{\mathcal{M}} \circ \mathcal{I}_G \circ \Phi)|_{V^M} = (\mathcal{I}_{\mathbf{H}} \circ \gamma_{\text{rad}} \circ \Phi)|_{V^M}$. But this implies $\gamma_{\mathcal{M}} \circ \mathcal{I}_G \circ \Phi = \mathcal{I}_{\mathbf{H}} \circ \gamma_{\text{rad}} \circ \Phi$ because of (8.4) for $\gamma_{\mathcal{M}}$ and γ_{rad} . Furthermore from (8.2) and (8.3) for $\gamma_{\mathcal{M}}$ and γ_{rad} , we have for $D \in U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})$ and $v \in V$

$$\begin{aligned} (\gamma_{\mathcal{M}} \circ \mathcal{I}_G)(D\Phi[v]) &= \gamma(D)(\gamma_{\mathcal{M}} \circ \mathcal{I}_G)(\Phi[v]) \\ &= \gamma(D)(\mathcal{I}_{\mathbf{H}} \circ \gamma_{\text{rad}})(\Phi[v]) = (\mathcal{I}_{\mathbf{H}} \circ \gamma_{\text{rad}})(D\Phi[v]). \end{aligned}$$

Now (17.2) follows since

$$\begin{aligned} (17.3) \quad \Xi_{\text{rad}}(\mathcal{X}) &= \sum_{V \in \widehat{K}_M} \bar{P}_G(V) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X}) \\ &= \sum_{V \in \widehat{K}_M} U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})((1 \otimes V) \otimes \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X})) \\ &= \sum_{V \in \widehat{K}_M} \sum_{\varphi \in \text{Hom}_{\mathbf{H}}(P_{\mathbf{H}}(V_{\text{single}}^M), \mathcal{X})} U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})((I_V \otimes \varphi)[V]) \\ &= \sum_{V \in \widehat{K}_M} \sum_{\Phi \in \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{rad}}(\mathcal{X}))} U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}})\Phi[V]. \end{aligned}$$

Here $I_V \in \text{Hom}_K(V, \bar{P}_G(V))$ is the map $V \ni v \mapsto 1 \otimes v \in \bar{P}_G(V)$.

Secondly for any $V \in \widehat{K}_M$ and $\Phi \in \text{Hom}_K(V, \Xi_{\text{rad}}(\mathcal{X}))$ we have

$$\begin{aligned} \Phi \in \text{Hom}_K(V, \text{Ker } \mathcal{I}_G) &\iff \mathcal{I}_G \circ \Phi = 0 \quad (\in \text{Hom}_K(V, \mathcal{M}_G)) \\ &\iff (\gamma_{\mathcal{M}} \circ \mathcal{I}_G \circ \Phi)[V^M] = \{0\} \quad (\because \text{Remark 8.8}) \\ &\iff (\mathcal{I}_{\mathbf{H}} \circ \gamma_{\text{rad}} \circ \Phi)[V^M] = \{0\} \quad (\because (17.2)) \end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow \gamma_{\text{rad}}(\Phi[V^M]) \subset \text{Ker } \mathcal{I}_{\mathbf{H}} \\
&\Longleftrightarrow \gamma_{\text{rad}}(\Phi[V]) \subset \text{Ker } \mathcal{I}_{\mathbf{H}} \quad (\because (8.4) \text{ for } \gamma_{\text{rad}}) \\
&\Longleftrightarrow \Phi[V] \subset \Xi_{(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})}^{\natural}(\text{Ker } \mathcal{I}_{\mathbf{H}}) \\
&\Longleftrightarrow \Phi \in \text{Hom}_K(V, \Xi_{(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})}^{\natural}(\text{Ker } \mathcal{I}_{\mathbf{H}})).
\end{aligned}$$

This shows (17.1). \square

Theorem 17.3. *Both $\mathbf{H}\text{-Mod} \ni \mathcal{X} \mapsto \Xi^{\min}(\mathcal{X}) \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ and $\mathbf{H}\text{-Mod} \ni \mathcal{X} \mapsto (\Xi^{\min}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$ are functors. If $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{rad}}$ has a radial restriction $\gamma_{\mathcal{M}}$ satisfying (rest-3) and if $\mathcal{I}_{\mathbf{H}} : \mathcal{X} \rightarrow \mathcal{M}_{\mathbf{H}}$ is an \mathbf{H} -homomorphism, then there exists a unique $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I}_G : \Xi^{\min}(\mathcal{X}) \rightarrow \mathcal{M}_G$ such that $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : (\Xi^{\min}(\mathcal{X}), \mathcal{X}) \rightarrow (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ is a morphism of \mathcal{C}_{rad} . Here it holds that*

$$\gamma_{\mathcal{M}} \circ \mathcal{I}_G = \mathcal{I}_{\mathbf{H}} \circ \gamma^{\min}.$$

Furthermore, if $\mathcal{I}_{\mathbf{H}}$ is injective then $\Xi^{\min}(\mathcal{X})$ is isomorphic to $\Xi_{\mathcal{M}}^{\min}(\text{Im } \mathcal{I}_{\mathbf{H}})$ by \mathcal{I}_G and hence $(\Xi^{\min}(\mathcal{X}), \mathcal{X}) \simeq (\Xi_{\mathcal{M}}^{\min}(\text{Im } \mathcal{I}_{\mathbf{H}}), \text{Im } \mathcal{I}_{\mathbf{H}})$ as a radial pair.

Proof. All except the last statement easily follow from Theorem 16.4, Lemma 17.2 and (17.2). Assume $\mathcal{I}_{\mathbf{H}}$ is injective. Then \mathcal{I}_G is also injective by Lemma 17.2. On the other hand, from (17.3) and (8.15) we can see $\Xi_{(\Xi^{\min}(\mathcal{X}), \mathcal{X})}^{\min}(\mathcal{X}) = \Xi^{\min}(\mathcal{X})$. Using (8.15) again we get

$$\text{Im } \mathcal{I}_G = \mathcal{I}_G(\Xi_{(\Xi^{\min}(\mathcal{X}), \mathcal{X})}^{\min}(\mathcal{X})) = \Xi_{\mathcal{M}}^{\min}(\text{Im } \mathcal{I}_{\mathbf{H}}).$$

Thus the theorem is proved. \square

Corollary 17.4. *If $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{rad}}$ has a radial restriction $\gamma_{\mathcal{M}}$ satisfying (rest-3) then the functor $\Xi^{\min} : \mathbf{H}\text{-Mod} \rightarrow (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ extends the correspondence*

$$\Xi_{\mathcal{M}}^{\min} : \{\mathbf{H}\text{-submodules of } \mathcal{M}_{\mathbf{H}}\} \rightarrow \{(\mathfrak{g}_{\mathbb{C}}, K)\text{-submodules of } \mathcal{M}_G\}$$

defined in Definition 8.21.

For example $(P_G(\mathbb{C}_{\text{triv}}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}))$ and $(P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}))$ with $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfy the assumption of the corollary (cf. Example 8.15 and Proposition 12.8). Since $\Xi^{\min}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}})) = P_G(\mathbb{C}_{\text{triv}})$, we have $\Xi_{(P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}), P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda}))}^{\min}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})) = P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda})$ by (8.15). Hence

$$(17.4) \quad \Xi^{\min}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \bar{\lambda})) = P_G(\mathbb{C}_{\text{triv}}, \bar{\lambda}) \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*).$$

From Proposition 16.8 and Lemma 17.2 we also have the following:

Corollary 17.5. *Suppose*

$$0 \longrightarrow \mathcal{X}_1 \longrightarrow \mathcal{X}_2 \longrightarrow \mathcal{X}_3 \longrightarrow 0$$

is an exact sequence in $\mathbf{H}\text{-Mod}$. Then in $(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ both

$$0 \longrightarrow \Xi^{\min}(\mathcal{X}_1) \longrightarrow \Xi^{\min}(\mathcal{X}_2) \quad \text{and} \quad \Xi^{\min}(\mathcal{X}_2) \longrightarrow \Xi^{\min}(\mathcal{X}_3) \longrightarrow 0$$

are exact.

The functor Ξ^{\min} has some better properties when restricted to $\mathbf{H}\text{-Mod}^{\text{fd}}$. Suppose $\mathcal{X} \in \mathbf{H}\text{-Mod}^{\text{fd}}$. Then $\Xi^{\min}(\mathcal{X}) \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}^{\text{fl}}$ by Theorem 16.7. First we shall give another realization of $\Xi^{\min}(\mathcal{X})$ which is useful to deduce some properties. Let σ denote the action of \mathbf{H} on \mathcal{X} . Since \mathcal{X} has finite dimension, σ can be integrated to the action of the simply connected Lie group A :

$$A \ni a \longmapsto a^\sigma := \exp \sigma(\log a) \in \text{End}_{\mathbb{C}} \mathcal{X}.$$

Let $\text{Ind}_{MAN}^G(\mathcal{X})$ be as in Definition 9.1. Let $\gamma_{\text{Ind}} : \text{Ind}_{MAN}^G(\mathcal{X}) \rightarrow \mathcal{X}$ be the linear map defined by $F(g) \mapsto F(1)$. (Note this is different from the map $\gamma_{\text{Ind}(\mathcal{X})} : \text{Ind}_{MAN}^G(\mathcal{X}) \rightarrow \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X})$ in Definition 9.7.) One easily observes this satisfies (rest-3). For each $V \in \widehat{K}_M$ we have the natural identification

$$(17.5) \quad \begin{aligned} & \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{X})) \simeq \text{Hom}_{\mathbb{C}}(V^M, \mathcal{X}); \\ & \Phi \longmapsto (\varphi : V^M \hookrightarrow V \xrightarrow{\Phi} \text{Ind}_{MAN}^G(\mathcal{X}) \xrightarrow{\gamma_{\text{Ind}}} \mathcal{X}) \end{aligned}$$

(cf. the proof of Theorem 9.8). Identify $V \otimes \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{X}))$ with the V -isotypic component of $\text{Ind}_{MAN}^G(\mathcal{X})$. We also identify the linear subspace

$$\{\varphi \in \text{Hom}_{\mathbb{C}}(V^M, \mathcal{X}); \varphi[V_{\text{double}}^M] = \{0\}\} \subset \text{Hom}_{\mathbb{C}}(V^M, \mathcal{X})$$

with $\text{Hom}_{\mathbb{C}}(V_{\text{single}}^M, \mathcal{X})$. Hence we can consider

$$\begin{aligned} V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) &\subset V \otimes \text{Hom}_{\mathbb{C}}(V_{\text{single}}^M, \mathcal{X}) \subset V \otimes \text{Hom}_{\mathbb{C}}(V^M, \mathcal{X}) \\ &\simeq V \otimes \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{X})) \subset \text{Ind}_{MAN}^G(\mathcal{X})_{K\text{-finite}}. \end{aligned}$$

This induces a natural $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism

$$\mathcal{I}_G : \Xi_{\text{w-rad}}(\mathcal{X}) = \bigoplus_{V \in \widehat{K}_M} \bar{P}_G(V) \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) \longrightarrow \text{Ind}_{MAN}^G(\mathcal{X})_{K\text{-finite}};$$

$$D \otimes v \otimes \varphi \longmapsto D\Phi[v] \quad \text{for} \quad \begin{cases} D \in U(\mathfrak{g}_{\mathbb{C}}), v \in V \text{ and} \\ \varphi \in \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) \\ \text{corresponding to} \\ \Phi \in \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{X})). \end{cases}$$

We put $\Xi_{\text{Ind}}(\mathcal{X}) = \text{Im } \mathcal{I}_G$. Note that $\Xi_{\text{Ind}}(\mathcal{X})$ is the submodule of $\text{Ind}_{MAN}^G(\mathcal{X})$ generated by

$$\bigcup_{V \in \widehat{K}_M} V \otimes \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}).$$

Theorem 17.6. *For any $\mathcal{X} \in \mathbf{H}\text{-Mod}^{\text{fd}}$, $(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X})$ is a radial pair and the linear map γ_{Ind} is its radial restriction satisfying (rest-3). Furthermore $(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X}) \simeq (\Xi^{\min}(\mathcal{X}), \mathcal{X})$ as a radial pair.*

Proof. First we assert $\gamma_{\text{w-rad}} : \Xi_{\text{w-rad}}(\mathcal{X}) \rightarrow \mathcal{X}$ coincides with $\gamma_{\text{Ind}} \circ \mathcal{I}_G$. Indeed $\Xi_{\text{w-rad}}(\mathcal{X})$ is spanned by

$$\{Df(\lambda) \otimes v \otimes \varphi; D \in U(\mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}}), v \in V, \varphi \in \text{Hom}_W(V_{\text{single}}^M, \mathcal{X})\}$$

over \mathbb{C} and we have

$$\begin{aligned}
\gamma_{\text{Ind}}(\mathcal{I}_G(D \otimes v \otimes \varphi)) &= \gamma_{\text{Ind}}(D\Phi[v]) \quad \text{with } \Phi \in \text{Hom}_K(V, \text{Ind}_{MAN}^G(\mathcal{X})) \\
&\hspace{15em} \text{corresponding to } \varphi \\
&= \gamma(D)(\Phi[v](1)) \\
&= \gamma(D)\varphi[p^V(v)] \\
&= \gamma_{\text{w-rad}}(D \otimes v \otimes \varphi)
\end{aligned}$$

where p^V denotes the orthogonal projection $V \rightarrow V^M$.

Now for $V \in \widehat{K}_M$ put

$$(17.6) \quad \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{Ind}}(\mathcal{X})) = \{ \Phi \in \text{Hom}_K(V, \Xi_{\text{Ind}}(\mathcal{X})); (\gamma_{\text{Ind}} \circ \Phi)[V_{\text{double}}^M] = \{0\} \}$$

and define the linear map $\tilde{\Gamma}_{\text{Ind}}^V : \text{Hom}_K(V, \Xi_{\text{Ind}}(\mathcal{X})) \rightarrow \text{Hom}_{\mathbb{C}}(V_{\text{single}}^M, \mathcal{X})$ by

$$(17.7) \quad \Phi \longmapsto (V_{\text{single}}^M \hookrightarrow V \xrightarrow{\Phi} \Xi_{\text{Ind}}(\mathcal{X}) \xrightarrow{\gamma_{\text{Ind}}} \mathcal{X}).$$

Let us prove $(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{Ch}}$ by these data. First it follows from the first assertion proved above and (rest-2) for $\gamma_{\text{w-rad}}$ that $\tilde{\Gamma}_{\text{w-rad}}^V = \tilde{\Gamma}_{\text{Ind}}^V \circ (\mathcal{I}_G \circ \cdot)$. Hence by the surjectivity of $\mathcal{I}_G \circ \cdot : \text{Hom}_K(V, \Xi_{\text{w-rad}}(\mathcal{X})) \rightarrow \text{Hom}_K(V, \Xi_{\text{Ind}}(\mathcal{X}))$ we have

$$\text{Im } \tilde{\Gamma}_{\text{Ind}}^V = \text{Im } \tilde{\Gamma}_{\text{w-rad}}^V = \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}).$$

Hence the restriction of (17.5) to $\text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{Ind}}(\mathcal{X}))$ reduces to

$$(17.8) \quad \tilde{\Gamma}_{\text{Ind}}^V : \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\text{Ind}}(\mathcal{X})) \xrightarrow{\sim} \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}).$$

Thus $(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{Ch}}$.

Now by (17.6) and (17.7), γ_{Ind} satisfies (rest-1) and (rest-2) in addition to (rest-3). Hence it follows from Lemma 8.13 that $(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X}) \in \mathcal{C}_{\text{rad}}$. Finally since it is clear by (17.8) and the definition of $\Xi_{\text{Ind}}(\mathcal{X})$ that $\Xi_{(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X})}^{\min}(\mathcal{X}) = \Xi_{\text{Ind}}(\mathcal{X})$, it follows from Theorem 17.3 that $(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X}) \simeq (\Xi^{\min}(\mathcal{X}), \mathcal{X})$. \square

From this realization we can deduce a double induction type property of Ξ^{\min} , which however we hope to discuss in a subsequent paper. Another application is lifting of a sesquilinear pairing in $\mathbf{H}\text{-Mod}^{\text{fd}}$ by Ξ^{\min} . Suppose $\mathcal{X} \in \mathbf{H}\text{-Mod}^{\text{fd}}$. Let \mathcal{X}^* be the linear space of antilinear functionals on \mathcal{X} and $(\cdot, \cdot)_{\mathcal{X}}$ the canonical sesquilinear form on $\mathcal{X}^* \times \mathcal{X}$. Then \mathcal{X}^* is naturally an \mathbf{H} -module by

$$(hx^*, x)_{\mathcal{X}} = (x^*, h^*x)_{\mathcal{X}} \quad \text{for } h \in \mathbf{H}, x^* \in \mathcal{X}^* \text{ and } x \in \mathcal{X}.$$

In particular $\mathfrak{a}_{\mathbb{C}} \subset \mathbf{H}$ acts on \mathcal{X}^* . But we strongly remark that this action is different from the $\mathfrak{a}_{\mathbb{C}}$ -action defined in the beginning of §9. To distinguish them, we use the symbol $\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*$ which stands for the linear space \mathcal{X}^* with the $\mathfrak{a}_{\mathbb{C}}$ -module structure given by

$$(\xi x^*, x)_{\mathcal{X}} = -(x^*, \xi x)_{\mathcal{X}} \quad \text{for } \xi \in \mathfrak{a}, x^* \in \mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^* \text{ and } x \in \mathcal{X}.$$

Let $\mathcal{X}(\cdot, \cdot)_{\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*}^{\mathbf{H}}$ be the sesquilinear form on $\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}) \times \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)$ defined in Definition 9.3 and let us consider the injective \mathbf{H} -homomorphism

$$\iota : \mathcal{X} \ni x \longmapsto (\mathbf{H} \ni h \longmapsto {}^t h x \in \mathcal{X}) \in \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}).$$

Then by Proposition 9.5 there exists a surjective \mathbf{H} -homomorphism $\iota^* : \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*) \rightarrow \mathcal{X}^*$ such that

$$(17.9) \quad \mathcal{X}(\iota(x), F)_{\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*}^{\mathbf{H}} = (x, \iota^*(F))_{\mathcal{X}^*} \quad \text{for } x \in \mathcal{X} \text{ and } F \in \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)$$

where $(x, x^*)_{\mathcal{X}^*} = \overline{(x^*, x)}_{\mathcal{X}}$. Applying Ξ_{Ind} to these morphisms we obtain

$$\begin{aligned} \Xi_{\text{Ind}}(\iota) : \Xi_{\text{Ind}}(\mathcal{X}) &\hookrightarrow \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X})), \\ \Xi_{\text{Ind}}(\iota^*) : \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)) &\twoheadrightarrow \Xi_{\text{Ind}}(\mathcal{X}^*). \end{aligned}$$

Note $\Xi_{\text{Ind}}(\iota^*)$ is surjective by Corollary 17.5. On the other hand, the $\mathfrak{a}_{\mathbb{C}}$ -homomorphisms

$$\begin{aligned} \text{ev} : \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}) \ni F(h) &\longmapsto F(1) \in \mathcal{X}, \\ \text{ev}' : \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*) \ni F(h) &\longmapsto F(1) \in \mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^* \end{aligned}$$

induce $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphisms

$$\begin{aligned} \beta : \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X})) &\hookrightarrow \text{Ind}_{MAN}^G(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}))_{K\text{-finite}} \longrightarrow \text{Ind}_{MAN}^G(\mathcal{X})_{K\text{-finite}}, \\ \beta' : \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)) &\hookrightarrow \text{Ind}_{MAN}^G(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*))_{K\text{-finite}} \longrightarrow \text{Ind}_{MAN}^G(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)_{K\text{-finite}}. \end{aligned}$$

Let $\mathcal{X}(\cdot, \cdot)_{\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*}^G$ be the sesquilinear form on $\text{Ind}_{MAN}^G(\mathcal{X}) \times \text{Ind}_{MAN}^G(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)$ defined in Definition 9.1. Now we define the invariant sesquilinear form $(\cdot, \cdot)'$ on $\Xi_{\text{Ind}}(\mathcal{X}) \times \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*))$ by

$$(F_1, F_2)' = \mathcal{X}((\beta \circ \Xi_{\text{Ind}}(\iota))(F_1), \beta'(F_2))_{\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*}^G.$$

Lemma 17.7. *It holds that*

$$(17.10) \quad \begin{aligned} \{F_2 \in \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)); (F_1, F_2)' = 0 \text{ for any } F_1 \in \Xi_{\text{Ind}}(\mathcal{X})\} \\ = \Xi_{\text{Ind}}(\iota^*)^{-1}(\Xi_{(\Xi_{\text{Ind}}(\mathcal{X}^*), \mathcal{X}^*)}^{\max}(\{0\})) \supset \text{Ker } \Xi_{\text{Ind}}(\iota^*). \end{aligned}$$

Hence $(\cdot, \cdot)'$ induces an invariant sesquilinear form (\cdot, \cdot) on $\Xi_{\text{Ind}}(\mathcal{X}) \times \Xi_{\text{Ind}}(\mathcal{X}^*)$. Furthermore the pair of (\cdot, \cdot) and $(\cdot, \cdot)_{\mathcal{X}^*}$ is compatible with restriction in the sense of Definition 9.9.

Proof. Suppose $V \in \widehat{K}_M$ and let $\{v_1, \dots, v_{m'}\}$, $\{v_{m'+1}, \dots, v_m\}$ and $\{v_{m+1}, \dots, v_n\}$ be bases of V_{single}^M , V_{double}^M and $(V^M)^\perp$. Let $\{v_i^*\} \subset V^*$ be as in Definition 9.9. Let γ_{Ind} , γ'_{Ind} , γ''_{Ind} and γ'''_{Ind} be the canonical restriction maps for $(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X})$, $(\Xi_{\text{Ind}}(\mathcal{X}^*), \mathcal{X}^*)$, $(\Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X})), \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}))$ and $(\Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)), \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*))$ respectively.

Suppose $\Phi_1 \in \text{Hom}_K(V, \Xi_{\text{Ind}}(\mathcal{X}))$, $\Phi_2 \in \text{Hom}_K(V^*, \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}^{\star})))$. Since the diagram

$$\begin{array}{ccccc} \Xi_{\text{Ind}}(\mathcal{X}) & \xrightarrow{\Xi_{\text{Ind}}(\iota)} & \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X})) & \xrightarrow{\beta} & \text{Ind}_{MAN}^G(\mathcal{X})_{K\text{-finite}} \\ \gamma_{\text{Ind}} \downarrow & & \gamma''_{\text{Ind}} \downarrow & & \downarrow \text{evaluation at 1} \\ \mathcal{X} & \xrightarrow{\iota} & \text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}) & \xrightarrow{\text{ev}} & \mathcal{X} \end{array}$$

commutes,

$$(\beta \circ \Xi_{\text{Ind}}(\iota) \circ \Phi_1)[v](1) = (\text{ev} \circ \iota \circ \gamma_{\text{Ind}} \circ \Phi_1)[v] = (\iota \circ \gamma_{\text{Ind}} \circ \Phi_1)[v](1) \quad \text{for } v \in V.$$

Likewise we have

$$(\beta' \circ \Phi_2)[v^*](1) = (\text{ev}' \circ \gamma'''_{\text{Ind}} \circ \Phi_2)[v^*] = (\gamma'''_{\text{Ind}} \circ \Phi_2)[v^*](1) \quad \text{for } v^* \in V^*.$$

Hence in a similar way to the proof of Proposition 9.10 we calculate

$$\begin{aligned} \sum_{i=1}^n (\Phi_1[v_i], \Phi_2[v_i^*])' &= \sum_{i=1}^n \mathcal{X}((\beta \circ \Xi_{\text{Ind}}(\iota) \circ \Phi_1)[v_i], (\beta' \circ \Phi_2)[v_i^*])_{\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^G} \\ &= \sum_{i=1}^n \int_K ((\beta \circ \Xi_{\text{Ind}}(\iota) \circ \Phi_1)[v_i](k), (\beta' \circ \Phi_2)[v_i^*](k))_{\mathcal{X}^*} dk \\ &= \sum_{i=1}^n ((\beta \circ \Xi_{\text{Ind}}(\iota) \circ \Phi_1)[v_i](1), (\beta' \circ \Phi_2)[v_i^*](1))_{\mathcal{X}^*} \\ &= \sum_{i=1}^m ((\iota \circ \gamma_{\text{Ind}} \circ \Phi_1)[v_i](1), (\gamma'''_{\text{Ind}} \circ \Phi_2)[v_i^*](1))_{\mathcal{X}^*} \\ &= \sum_{i=1}^m \frac{1}{|W|} \sum_{w \in W} ((\iota \circ \gamma_{\text{Ind}} \circ \Phi_1)[w^{-1}v_i](1), (\gamma'''_{\text{Ind}} \circ \Phi_2)[w^{-1}v_i^*](1))_{\mathcal{X}^*}. \end{aligned}$$

Now suppose $\Phi_1 \in \text{Hom}_K^{2 \rightarrow 2}$ or $\Phi_2 \in \text{Hom}_K^{2 \rightarrow 2}$. Then $(\gamma_{\text{Ind}} \circ \Phi_1)|_{V_{\text{double}}^M} = 0$ or $(\gamma'''_{\text{Ind}} \circ \Phi_2)|_{V_{\text{double}}^M} = 0$. Since $\tilde{\Gamma}_{\text{Ind}}^V(\Phi_1) = (\gamma_{\text{Ind}} \circ \Phi_1)|_{V_{\text{single}}^M} \in \text{Hom}_W$ and $(\gamma'''_{\text{Ind}} \circ \Phi_2)|_{(V^*)_{\text{single}}^M} \in \text{Hom}_W$, the last expression above still equals

$$\begin{aligned} &\sum_{i=1}^{m'} \frac{1}{|W|} \sum_{w \in W} ((\iota \circ \gamma_{\text{Ind}} \circ \Phi_1)[w^{-1}v_i](1), (\gamma'''_{\text{Ind}} \circ \Phi_2)[w^{-1}v_i^*](1))_{\mathcal{X}^*} \\ &= \sum_{i=1}^{m'} \frac{1}{|W|} \sum_{w \in W} ((\iota \circ \tilde{\Gamma}_{\text{Ind}}^V(\Phi_1))[v_i](w), (\gamma'''_{\text{Ind}} \circ \Phi_2)[v_i^*](w))_{\mathcal{X}^*} \\ &= \sum_{i=1}^{m'} \mathcal{X}((\iota \circ \tilde{\Gamma}_{\text{Ind}}^V(\Phi_1))[v_i], (\gamma'''_{\text{Ind}} \circ \Phi_2)[v_i^*])_{\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^{\mathbf{H}}} \\ &= \sum_{i=1}^{m'} (\tilde{\Gamma}_{\text{Ind}}^V(\Phi_1)[v_i], (\iota^* \circ \gamma'''_{\text{Ind}} \circ \Phi_2)[v_i^*])_{\mathcal{X}^*} \quad (\because (17.9)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{m'} (\tilde{\Gamma}_{\text{Ind}}^V(\Phi_1)[v_i], (\gamma'_{\text{Ind}} \circ \Xi_{\text{Ind}}(\iota^*) \circ \Phi_2)[v_i^*])_{\mathcal{X}^*} \\
&= \sum_{i=1}^{m'} (\tilde{\Gamma}_{\text{Ind}}^V(\Phi_1)[v_i], \tilde{\Gamma}_{\text{Ind}}^{V^*}(\Xi_{\text{Ind}}(\iota^*) \circ \Phi_2)[v_i^*])_{\mathcal{X}^*}
\end{aligned}$$

where $\tilde{\Gamma}_{\text{Ind}}^{V^*}(\Phi) := (\gamma'_{\text{Ind}} \circ \Phi)|_{(V^*)_{\text{single}}^M}$ for $\Phi \in \text{Hom}_K(V^*, \Xi_{\text{Ind}}(\mathcal{X}^*))$. Thus if $\Phi_1 \in \text{Hom}_K^{2 \rightarrow 2}$ or $\Phi_2 \in \text{Hom}_K^{2 \rightarrow 2}$ then

$$(17.11) \quad \sum_{i=1}^n (\Phi_1[v_i], \Phi_2[v_i^*])' = \sum_{i=1}^{m'} (\tilde{\Gamma}_{\text{Ind}}^V(\Phi_1)[v_i], \tilde{\Gamma}_{\text{Ind}}^{V^*}(\Xi_{\text{Ind}}(\iota^*) \circ \Phi_2)[v_i^*])_{\mathcal{X}^*}.$$

Now denote the leftmost part of (17.10) by $\Xi_{\text{Ind}}(\mathcal{X})^\perp$. Suppose $E \in \hat{K}_M$ and take bases $\{e_1, \dots, e_\nu\} \subset E$ and $\{e_j^*\} \subset E^*$ as in Proposition 10.3. Since $\text{Hom}_K(E, \Xi_{\text{Ind}}(\mathcal{X}))$ is spanned over \mathbb{C} by elements of the form

$$\Phi_1 \circ \Psi \quad \text{with } V \in \hat{K}_M, \Phi_1 \in \text{Hom}_{\mathfrak{g}_{\mathbb{C}}, K}^{2 \rightarrow 2}(P_G(V), \Xi_{\text{Ind}}(\mathcal{X})) \text{ and } \Psi \in \text{Hom}_K(E, P_G(V)),$$

we have for $\Phi \in \text{Hom}_K(E^*, \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*)))$

$$\begin{aligned}
&\Phi \in \text{Hom}_K(E^*, \Xi_{\text{Ind}}(\mathcal{X})^\perp) \\
&\iff \sum_{j=1}^{\nu} ((\Phi_1 \circ \Psi)[e_j], \Phi[e_j^*])' = 0 \quad \forall V, \forall \Phi_1, \forall \Psi \\
&\iff \sum_{i=1}^n (\Phi_1[v_i], (\Phi \circ \Psi^*)[v_i^*])' = 0 \quad \forall V, \forall \Phi_1, \forall \Psi \quad (\because \text{Corollary 10.5}) \\
&\iff \sum_{i=1}^{m'} (\tilde{\Gamma}_{\text{Ind}}^V(\Phi_1)[v_i], \tilde{\Gamma}_{\text{Ind}}^{V^*}(\Xi_{\text{Ind}}(\iota^*) \circ \Phi \circ \Psi^*)[v_i^*])_{\mathcal{X}^*} = 0 \quad \forall V, \forall \Phi_1, \forall \Psi \quad (\because (17.11)) \\
&\iff \sum_{i=1}^{m'} (\varphi_1[v_i], \tilde{\Gamma}_{\text{Ind}}^{V^*}(\Xi_{\text{Ind}}(\iota^*) \circ \Phi \circ \Psi^*)[v_i^*])_{\mathcal{X}^*} = 0 \quad \forall V, \forall \Psi, \forall \varphi_1 \in \text{Hom}_W(V_{\text{single}}^M, \mathcal{X}) \\
&\iff \tilde{\Gamma}_{\text{Ind}}^{V^*}(\Xi_{\text{Ind}}(\iota^*) \circ \Phi \circ \Psi^*) = 0 \quad \forall V, \forall \Psi \\
&\iff \tilde{\Gamma}_{\text{Ind}}^{V^*}(\Phi_2) = 0 \quad \forall V, \forall \Phi_2 \in \text{Hom}_K(V^*, U(\mathfrak{g}_{\mathbb{C}})(\Xi_{\text{Ind}}(\iota^*) \circ \Phi)[E^*]) \\
&\iff U(\mathfrak{g}_{\mathbb{C}})(\Xi_{\text{Ind}}(\iota^*) \circ \Phi)[E^*] \subset \Xi_{(\Xi_{\text{Ind}}(\mathcal{X}^*), \mathcal{X}^*)}^{\max}(\{0\}) \\
&\iff \Xi_{\text{Ind}}(\iota^*) \circ \Phi \in \text{Hom}_K(E^*, \Xi_{(\Xi_{\text{Ind}}(\mathcal{X}^*), \mathcal{X}^*)}^{\max}(\{0\})).
\end{aligned}$$

Thus we get (17.10) and the induced form (\cdot, \cdot) on $\Xi_{\text{Ind}}(\mathcal{X}) \times \Xi_{\text{Ind}}(\mathcal{X}^*)$.

The compatibility with restriction of the pair of (\cdot, \cdot) and $(\cdot, \cdot)_{\mathcal{X}^*}$ follows from (17.11) and the surjectivity of the following maps:

$$\begin{aligned}
&\text{Hom}_K(V^*, \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*))) \xrightarrow{\Xi_{\text{Ind}}(\iota^*) \circ \cdot} \text{Hom}_K(V^*, \Xi_{\text{Ind}}(\mathcal{X}^*)), \\
&\text{Hom}_K^{2 \rightarrow 2}(V^*, \Xi_{\text{Ind}}(\text{Ind}_{S(\mathfrak{a}_{\mathbb{C}})}^{\mathbf{H}}(\mathcal{X}_{\mathfrak{a}_{\mathbb{C}}}^*))) \xrightarrow{\Xi_{\text{Ind}}(\iota^*) \circ \cdot} \text{Hom}_K^{2 \rightarrow 2}(V^*, \Xi_{\text{Ind}}(\mathcal{X}^*)). \quad \square
\end{aligned}$$

Theorem 17.8. *Suppose $\mathcal{X}_1, \mathcal{X}_2 \in \mathbf{H}\text{-Mod}^{\text{fd}}$ and let $(\cdot, \cdot)^{\mathbf{H}}$ be an invariant sesquilinear form on $\mathcal{X}_1 \times \mathcal{X}_2$. Then there exists a unique invariant sesquilinear form $(\cdot, \cdot)^G$ on $\Xi^{\min}(\mathcal{X}_1) \times \Xi^{\min}(\mathcal{X}_2)$ such that the pair $(\cdot, \cdot)^G$ and $(\cdot, \cdot)^{\mathbf{H}}$ is compatible with restriction in the sense of Definition 9.9.*

Proof. Let $\mathcal{X}_1^* \in \mathbf{H}\text{-Mod}^{\text{fd}}$ be as before. Then the canonical sesquilinear form $(\cdot, \cdot)_{\mathcal{X}_1^*}$ on $\mathcal{X}_1 \times \mathcal{X}_1^*$ can be lifted to a sesquilinear form (\cdot, \cdot) on $\Xi^{\min}(\mathcal{X}_1) \times \Xi^{\min}(\mathcal{X}_1^*)$ by Lemma 17.7. Now there exists a unique \mathbf{H} -homomorphism $\mathcal{I}_{\mathbf{H}} : \mathcal{X}_2 \rightarrow \mathcal{X}_1^*$ such that

$$(x_1, x_2)^{\mathbf{H}} = (x_1, \mathcal{I}_{\mathbf{H}}(x_2))_{\mathcal{X}_1^*} \quad \text{for } x_1 \in \mathcal{X}_1 \text{ and } x_2 \in \mathcal{X}_2.$$

Using $\Xi^{\min}(\mathcal{I}_{\mathbf{H}}) : \Xi^{\min}(\mathcal{X}_2) \rightarrow \Xi^{\min}(\mathcal{X}_1^*)$ we define

$$(y_1, y_2)^G = (y_1, \Xi^{\min}(\mathcal{I}_{\mathbf{H}})(y_2)) \quad \text{for } y_1 \in \Xi^{\min}(\mathcal{X}_1) \text{ and } y_2 \in \Xi^{\min}(\mathcal{X}_2).$$

This is clearly an invariant sesquilinear form on $\Xi^{\min}(\mathcal{X}_1) \times \Xi^{\min}(\mathcal{X}_2)$ which, together with $(\cdot, \cdot)^{\mathbf{H}}$, is compatible with restriction. Such a sesquilinear form is unique by Corollary 10.6. \square

18. THE FUNCTOR Ξ

If $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \in \mathcal{C}_{\text{rad}}$ and $\mathcal{Y} \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ are given then we can make a new radial pair $\mathcal{M}' = (\mathcal{M}_G \times \mathcal{Y}, \mathcal{M}_{\mathbf{H}})$ by putting for each $V \in \hat{K}_M$

$$\begin{aligned} \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G \times \mathcal{Y}) &= \text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \times \{0\}, \\ \tilde{\Gamma}_{\mathcal{M}'}^V &= \tilde{\Gamma}_{\mathcal{M}}^V \circ (\text{Hom}_K(V, \mathcal{M}_G \times \mathcal{Y}) \xrightarrow{\text{projection}} \text{Hom}_K(V, \mathcal{M}_G)). \end{aligned}$$

Here we note $\Xi_{\mathcal{M}'}^{\min}(\mathcal{M}_{\mathbf{H}}) \subset \mathcal{M}_G \times \{0\}$ and $\Xi_{\mathcal{M}'}^{\max}(\{0\}) \supset \{0\} \times \mathcal{Y}$. This example shows a radial pair $(\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ may contain a redundant part which gives no link between \mathcal{M}_G and $\mathcal{M}_{\mathbf{H}}$.

Definition 18.1. We say a radial pair $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ is *reduced* if $\Xi_{\mathcal{M}}^{\min}(\mathcal{M}_{\mathbf{H}}) = \mathcal{M}_G$ and $\Xi_{\mathcal{M}}^{\max}(\{0\}) = \{0\}$.

Proposition 18.2. *For any \mathbf{H} -submodule \mathcal{X} of $\mathcal{A}(A)$, $(\Xi_0^{\min}(\mathcal{X}), \mathcal{X})$ is a reduced radial pair. Hence by Theorem 11.6 $(X_G(\lambda), X_{\mathbf{H}}(\lambda))$ is reduced for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.*

Proof. First, let $(\mathcal{I}_G, \mathcal{I}_{\mathbf{H}}) : (\Xi_0^{\min}(\mathcal{X}), \mathcal{X}) \rightarrow (\mathcal{A}(G/K)_{K\text{-finite}}, \mathcal{A}(A))$ be the pair of inclusions (cf. Corollary 15.5). Then by (8.15) we have

$$\Xi_{(\Xi_0^{\min}(\mathcal{X}), \mathcal{X})}^{\min}(\mathcal{X}) = \mathcal{I}_G(\Xi_{(\Xi_0^{\min}(\mathcal{X}), \mathcal{X})}^{\min}(\mathcal{X})) = \Xi_0^{\min}(\mathcal{I}_{\mathbf{H}}(\mathcal{X})) = \Xi_0^{\min}(\mathcal{X}).$$

Secondly, put $\mathcal{N} = \Xi_{(\Xi_0^{\min}(\mathcal{X}), \mathcal{X})}^{\max}(\{0\})$. Then from Theorem 8.23 (ii) we have

$$(18.1) \quad \tilde{\Gamma}_0^{\mathbb{C}_{\text{triv}}}(\text{Hom}_K(\mathbb{C}_{\text{triv}}, \mathcal{N})) = \text{Hom}_W(\mathbb{C}_{\text{triv}}, \{0\}) = \{0\}.$$

Assume now $\mathcal{N} \ni f \neq 0$. Since the sesquilinear form $(\cdot, \cdot)_r^G$ defined by (11.1) is non-degenerate on $\mathcal{A}(G/K)_{K\text{-finite}} \times P_G(\mathbb{C}_{\text{triv}})$, there exists some $D \in U(\mathfrak{g}_{\mathbb{C}})$ such that $(f, D \otimes v_{\text{triv}})_r^G \neq 0$. Thus $(D^*f, 1 \otimes v_{\text{triv}})_r^G \neq 0$. This means \mathcal{N} contains a non-zero K -invariant element, contrary to (18.1). Hence $\mathcal{N} = \{0\}$. \square

Suppose $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_H) \in \mathcal{C}_{\text{rad}}$ and put $\mathcal{M}' = (\Xi_{\mathcal{M}}^{\min}(\mathcal{M}_H), \mathcal{M}_H)$. Then it follows from Theorem 8.23 (ii) that \mathcal{M}' and $(\Xi_{\mathcal{M}'}^{\max}(\{0\}), \{0\})$ are radial pairs. Moreover as the cokernel of $(\Xi_{\mathcal{M}'}^{\max}(\{0\}), \{0\}) \hookrightarrow \mathcal{M}'$ we have $(\Xi_{\mathcal{M}}^{\min}(\mathcal{M}_H) / \Xi_{\mathcal{M}'}^{\max}(\{0\}), \mathcal{M}_H) \in \mathcal{C}_{\text{rad}}$ by Proposition 8.12 and this is reduced by Theorem 8.27. Furthermore for each $V \in \widehat{K}_M$ it naturally holds that

$$\text{Hom}_K^{2 \rightarrow 2}(V, \mathcal{M}_G) \simeq \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{M}_H)) \simeq \text{Hom}_K^{2 \rightarrow 2}(V, \Xi_{\mathcal{M}}^{\min}(\mathcal{M}_H) / \Xi_{\mathcal{M}'}^{\max}(\{0\})).$$

In this way we can always extract the reduced part from any radial pair.

Let us now construct a functor $\Xi : \mathbf{H}\text{-Mod} \rightarrow (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}$ such that $\mathcal{M}_G = \Xi(\mathcal{M}_H)$ for any reduced radial pair $(\mathcal{M}_G, \mathcal{M}_H)$. Throughout the paper we have given many examples of radial pairs. If we extract the reduced part $(\mathcal{M}_G, \mathcal{M}_H)$ from any such pair by the above method, then $\mathcal{M}_G = \Xi(\mathcal{M}_H)$.

Definition 18.3 (the functor Ξ). For any $\mathcal{X} \in \mathbf{H}\text{-Mod}$ we put

$$\Xi(\mathcal{X}) = \Xi_{\text{rad}}(\mathcal{X}) / \Xi_{(\Xi_{\text{rad}}(\mathcal{X}), \mathcal{X})}^{\max}(\{0\}).$$

Then $(\Xi(\mathcal{X}), \mathcal{X})$ is a reduced radial pair.

Theorem 18.4. (i) *If a reduced radial pair $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_H)$ and an \mathbf{H} -homomorphism $\mathcal{I}_H : \mathcal{X} \rightarrow \mathcal{M}_H$ are given, then there exists a unique $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I}_G : \Xi(\mathcal{X}) \rightarrow \mathcal{M}_G$ such that $(\mathcal{I}_G, \mathcal{I}_H) : (\Xi(\mathcal{X}), \mathcal{X}) \rightarrow (\mathcal{M}_G, \mathcal{M}_H)$ is a morphism of \mathcal{C}_{rad} . If \mathcal{I}_H is injective, so is \mathcal{I}_G . If \mathcal{I}_H is surjective, so is \mathcal{I}_G .*
(ii) *Suppose a radial pair $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_H)$ satisfies $\Xi_{\mathcal{M}}^{\min}(\mathcal{M}_H) = \mathcal{M}_G$ and $\mathcal{I}_H : \mathcal{M}_H \rightarrow \mathcal{X}$ is an \mathbf{H} -homomorphism. Then there exists a unique $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I}_G : \mathcal{M}_G \rightarrow \Xi(\mathcal{X})$ such that $(\mathcal{I}_G, \mathcal{I}_H) : (\mathcal{M}_G, \mathcal{M}_H) \rightarrow (\Xi(\mathcal{X}), \mathcal{X})$ is a morphism of \mathcal{C}_{rad} . If \mathcal{I}_H is surjective, so is \mathcal{I}_G .*

Proof. Let \mathcal{M} and \mathcal{I}_H be as in (i). The existence and uniqueness of $\mathcal{I}_G : \Xi(\mathcal{X}) \rightarrow \mathcal{M}_G$ follow from Theorem 16.4 and (8.16). The injectivity of \mathcal{I}_H implies that of \mathcal{I}_G by (8.16). The surjectivity of \mathcal{I}_H implies that of \mathcal{I}_G by (8.15).

Secondly let \mathcal{M} and \mathcal{I}_H be as in (ii). Then we have the exact sequence

$$0 \rightarrow (\Xi_{\mathcal{M}}^{\max}(\text{Ker } \mathcal{I}_H), \text{Ker } \mathcal{I}_H) \xrightarrow{\iota} (\mathcal{M}_G, \mathcal{M}_H) \xrightarrow{\pi} (\mathcal{M}_G / \Xi_{\mathcal{M}}^{\max}(\text{Ker } \mathcal{I}_H), \text{Coim } \mathcal{I}_H) \rightarrow 0.$$

We assert $\mathcal{M}' := (\mathcal{M}_G / \Xi_{\mathcal{M}}^{\max}(\text{Ker } \mathcal{I}_H), \text{Coim } \mathcal{I}_H)$ is reduced. Indeed $\Xi_{\mathcal{M}'}^{\min}(\text{Coim } \mathcal{I}_H) = \mathcal{M}_G / \Xi_{\mathcal{M}}^{\max}(\text{Ker } \mathcal{I}_H)$ because of (8.15). In addition, since $\pi = (\pi_G, \pi_H)$ is epi we have

$$\begin{aligned} \Xi_{\mathcal{M}'}^{\max}(\{0\}) &= \pi_G(\pi_G^{-1}(\Xi_{\mathcal{M}'}^{\max}(\{0\}))) \\ &= \pi_G(\Xi_{\mathcal{M}}^{\max}(\text{Ker } \pi_H)) & (\because (8.16)) \\ &= \pi_G(\Xi_{\mathcal{M}}^{\max}(\text{Ker } \mathcal{I}_H)) = \{0\}. \end{aligned}$$

Hence $\mathcal{M}' \simeq (\Xi(\text{Coim } \mathcal{I}_H), \text{Coim } \mathcal{I}_H)$ as a radial pair by (i). Using (i) again we can uniquely lift the \mathbf{H} -homomorphism $\mathcal{I}'_H : \text{Coim } \mathcal{I}_H \rightarrow \mathcal{X}$ with $\mathcal{I}_H = \mathcal{I}'_H \circ \pi_H$ to a morphism $\mathcal{I}' = (\mathcal{I}'_G, \mathcal{I}'_H) : \mathcal{M}' \rightarrow (\Xi(\mathcal{X}), \mathcal{X})$ of \mathcal{C}_{rad} . Hence if we put $\mathcal{I}_G = \mathcal{I}'_G \circ \pi_G$ then $(\mathcal{I}_G, \mathcal{I}_H) : (\mathcal{M}_G, \mathcal{M}_H) \rightarrow (\Xi(\mathcal{X}), \mathcal{X})$ is a morphism. We note \mathcal{I}'_G is surjective if

$\mathcal{I}'_{\mathbf{H}}$ is surjective. What remains to be shown is the uniqueness of \mathcal{I}_G . Assume $(\mathcal{I}'_G, \mathcal{I}_{\mathbf{H}}) : (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \rightarrow (\Xi(\mathcal{X}), \mathcal{X})$ is also a morphism. Then by (8.16) we have

$$\text{Ker } \mathcal{I}'_G = \mathcal{I}_G'^{-1}(\{0\}) = \mathcal{I}_G'^{-1}(\Xi_{(\Xi(\mathcal{X}), \mathcal{X})}^{\max}(\{0\})) = \Xi_{\mathcal{M}}^{\max}(\mathcal{I}_{\mathbf{H}}^{-1}(\{0\})) = \Xi_{\mathcal{M}}^{\max}(\text{Ker } \mathcal{I}_{\mathbf{H}}).$$

Hence $(\mathcal{I}'_G, \mathcal{I}_{\mathbf{H}})$ factors through π . From the uniqueness of \mathcal{I}'_G we conclude $\mathcal{I}'_G = \mathcal{I}_G \circ \pi_G = \mathcal{I}_G$. \square

Corollary 18.5. (i) For any $\mathcal{X} \in \mathbf{H}\text{-Mod}$

$$\Xi(\mathcal{X}) = \Xi^{\min}(\mathcal{X}) / \Xi_{(\Xi^{\min}(\mathcal{X}), \mathcal{X})}^{\max}(\{0\}).$$

(ii) Suppose a radial pair $\mathcal{M} = (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}})$ satisfies $\Xi_{\mathcal{M}}^{\min}(\mathcal{M}_{\mathbf{H}}) = \mathcal{M}_G$. Then the identity morphism on $\mathcal{M}_{\mathbf{H}}$ naturally induces two consecutive epimorphisms

$$(\Xi_{\text{rad}}(\mathcal{M}_{\mathbf{H}}), \mathcal{M}_{\mathbf{H}}) \twoheadrightarrow (\mathcal{M}_G, \mathcal{M}_{\mathbf{H}}) \twoheadrightarrow (\Xi(\mathcal{M}_{\mathbf{H}}), \mathcal{M}_{\mathbf{H}})$$

in \mathcal{C}_{rad} .

(iii) The functor Ξ extends the correspondence (15.5). In particular, $\Xi(X_{\mathbf{H}}(\lambda)) = X_G(\lambda)$ for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

In the below we shall see the functor Ξ commutes with conjugate dual operations. For any $\mathcal{X} \in \mathbf{H}\text{-Mod}^{\text{fd}}$ let \mathcal{X}^* and $(\cdot, \cdot)_{\mathcal{X}^*}$ be as in the last section. In general for $\mathcal{Y} \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}^{\text{fl}}$ we define $\mathcal{Y}^* \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}^{\text{fl}}$ as follows: For each $V \in \widehat{K}$ the V -isotypic component \mathcal{Y}_V of \mathcal{Y} has finite dimension; Put $\mathcal{Y}^* = \bigoplus_{V \in \widehat{K}} \mathcal{Y}_V^*$ where \mathcal{Y}_V^* is the space of antilinear functionals on \mathcal{Y}_V ; Using a natural non-degenerate sesquilinear form $(\cdot, \cdot)_{\mathcal{Y}}$ on $\mathcal{Y}^* \times \mathcal{Y}$, we define the $(\mathfrak{g}_{\mathbb{C}}, K)$ -module structure of \mathcal{Y}^* by

$$(Dy^*, y)_{\mathcal{Y}} = (y^*, D^*y)_{\mathcal{Y}}, \quad (ky^*, y)_{\mathcal{Y}} = (y^*, k^{-1}y)_{\mathcal{Y}}$$

for $D \in U(\mathfrak{g}_{\mathbb{C}})$, $k \in K$, $y^* \in \mathcal{Y}^*$ and $y \in \mathcal{Y}$; Then it is easy to see $\mathcal{Y}^* \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}^{\text{fl}}$. Since $\Xi^{\min}(\mathcal{X}) \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}^{\text{fl}}$ by Theorem 16.7, $\Xi^{\min}(\mathcal{X})^* \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}^{\text{fl}}$.

Theorem 18.6. In the setting above there exists a unique invariant sesquilinear form $(\cdot, \cdot)^G$ on $\Xi(\mathcal{X}) \times \Xi(\mathcal{X}^*)$ such that the pair $(\cdot, \cdot)^G$ and $(\cdot, \cdot)_{\mathcal{X}^*}$ is compatible with restriction in the sense of Definition 9.9. The form $(\cdot, \cdot)^G$ is non-degenerate. In particular $\Xi(\mathcal{X}^*) \simeq \Xi(\mathcal{X})^*$ as a $(\mathfrak{g}_{\mathbb{C}}, K)$ -module.

Proof. Let (\cdot, \cdot) be the sesquilinear form on $\Xi_{\text{Ind}}(\mathcal{X}) \times \Xi_{\text{Ind}}(\mathcal{X}^*)$ in Lemma 17.7. We note (17.10) can be rewritten as

$$\{F_2 \in \Xi_{\text{Ind}}(\mathcal{X}^*); (F_1, F_2) = 0 \text{ for any } F_1 \in \Xi_{\text{Ind}}(\mathcal{X})\} = \Xi_{(\Xi_{\text{Ind}}(\mathcal{X}^*), \mathcal{X}^*)}^{\max}(\{0\}).$$

We can interchange the roles of \mathcal{X} and \mathcal{X}^* by Corollary 10.6 to deduce

$$\{F_1 \in \Xi_{\text{Ind}}(\mathcal{X}); (F_1, F_2) = 0 \text{ for any } F_2 \in \Xi_{\text{Ind}}(\mathcal{X}^*)\} = \Xi_{(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X})}^{\max}(\{0\}).$$

Since

$$\Xi(\mathcal{X}) = \Xi_{\text{Ind}}(\mathcal{X}) / \Xi_{(\Xi_{\text{Ind}}(\mathcal{X}), \mathcal{X})}^{\max}(\{0\}), \quad \Xi(\mathcal{X}^*) = \Xi_{\text{Ind}}(\mathcal{X}^*) / \Xi_{(\Xi_{\text{Ind}}(\mathcal{X}^*), \mathcal{X}^*)}^{\max}(\{0\}),$$

(\cdot, \cdot) induces an invariant non-degenerate sesquilinear form $(\cdot, \cdot)^G$ on $\Xi(\mathcal{X}) \times \Xi(\mathcal{X}^*)$ which, together with $(\cdot, \cdot)_{\mathcal{X}^*}$, is compatible with restriction. The uniqueness follows from Corollary 10.6. \square

From this theorem one can deduce the following in the same way as Theorem 17.8:

Corollary 18.7. *If $\mathcal{X} \in \mathbf{H}\text{-Mod}^{\text{fd}}$ has a non-degenerate invariant Hermitian form $(\cdot, \cdot)^{\mathbf{H}}$ then there exists a unique non-degenerate invariant Hermitian form $(\cdot, \cdot)^G$ on $\Xi(\mathcal{X})$ such that the pair of $(\cdot, \cdot)^G$ and $(\cdot, \cdot)^{\mathbf{H}}$ is compatible with restriction.*

19. EXAMPLES FOR $G = SL(2, \mathbb{R})$

In this section we assume

$$G = SL(2, \mathbb{R}), \quad K = \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}; \varphi \in \mathbb{R} \right\}, \quad \mathfrak{s} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix}; a, b \in \mathbb{R} \right\}$$

and put

$$e_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_{\pm} = \frac{1}{2} \begin{pmatrix} 1 & \mp i \\ \mp i & -1 \end{pmatrix}.$$

Then the K -module $(\text{Ad}, \mathfrak{s}_{\mathbb{C}})$ has a unique irreducible decomposition

$$\mathfrak{s}_{\mathbb{C}} = \mathfrak{s}_+ \oplus \mathfrak{s}_- \quad \text{with} \quad \mathfrak{s}_{\pm} = \mathbb{C}e_{\pm}.$$

If we identify \widehat{K} with \mathbb{Z} by

$$\left(K \ni \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \mapsto e^{in\varphi} \in \mathbb{C}^{\times} \right) \longleftrightarrow n$$

then $\widehat{K}_M = 2\mathbb{Z}$ and $\mathfrak{s}_{\pm} \leftrightarrow \pm 2$. Furthermore, since G has real rank 1, it follows from [O, Corollary 2.9] that $\widehat{K}_{\text{qsp}} = \widehat{K}_{\text{sp}} = \{\mathbb{C}_{\text{triv}}, \mathfrak{s}_+, \mathfrak{s}_-\} = \{0, \pm 2\}$. Let $\alpha \in \mathfrak{a}^*$ be such that $\Sigma^+ = \{\alpha\}$ (and hence $R^+ = \{2\alpha\}$).

The classification of the irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules is classical (cf. [Kn]). Here we fix some notation. For $n = 1, 2, \dots$ we put

E_G^n = the irreducible representation with dimension n ,

$D_G^{n,+}$ = the discrete series representation with K -types $\{n+1, n+3, \dots\}$,

$D_G^{n,-}$ = the discrete series representation with K -types $\{-n-1, -n-3, \dots\}$.

If all the K -types of a given irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$ -module belong to \widehat{K}_M , then this module is equivalent to exactly one of the following:

$$\left\{ \begin{array}{l} B_G(\lambda)_{K\text{-finite}} \simeq B_G(-\lambda)_{K\text{-finite}} \text{ with } \lambda(\alpha^{\vee}) \notin \{\pm 1, \pm 3, \dots\}, \\ E_G^n \text{ with } n = 1, 3, \dots, \\ D_G^{n,+} \text{ with } n = 1, 3, \dots, \\ D_G^{n,-} \text{ with } n = 1, 3, \dots \end{array} \right.$$

It is also well known that for $n = 1, 3, \dots$, $B_G(\pm n\rho)_{K\text{-finite}}$ are indecomposable and

$$(19.1) \quad \begin{aligned} E_G^n &\subset B_G(n\rho)_{K\text{-finite}}, & B_G(n\rho)_{K\text{-finite}}/E_G^n &\simeq D_G^{n,+} \oplus D_G^{n,-}, \\ D_G^{n,+} \oplus D_G^{n,-} &\subset B_G(-n\rho)_{K\text{-finite}}, & B_G(-n\rho)_{K\text{-finite}}/(D_G^{n,+} \oplus D_G^{n,-}) &\simeq E_G^n. \end{aligned}$$

Hence from the bijectivity condition of \mathcal{P}_G^λ stated in Remark 12.2 (ii) we have

$$(19.2) \quad X_G(\lambda) \simeq \begin{cases} B_G(\lambda)_{K\text{-finite}} \simeq \mathcal{A}(G/K, \lambda)_{K\text{-finite}} & \text{if } \lambda(\alpha^\vee) \notin \{\pm 1, \pm 3, \dots\}, \\ E_G^{|\lambda(\alpha^\vee)|} & \text{if } \lambda(\alpha^\vee) \in \{\pm 1, \pm 3, \dots\}. \end{cases}$$

In addition, from Proposition 9.2 and Proposition 11.1 (ii) we have

$$(19.3) \quad P_G(\mathbb{C}_{\text{triv}}, \lambda) \simeq B_G(-\lambda)_{K\text{-finite}} \quad \text{for } \lambda(\alpha^\vee) \notin \{-1, -3, \dots\}.$$

Now let us look over the \mathbf{H} side. First $W = \{1, s_\alpha\}$ has only two irreducible modules, namely $\mathbb{C}_{\text{triv}} = \mathbb{C}v_{\text{triv}}$ and $\mathbb{C}_{\text{sgn}} = \mathbb{C}v_{\text{sgn}}$. If we define the $S(\mathfrak{a}_{\mathbb{C}})$ -action on them by

$$\xi v_{\text{triv}} = -\rho(\xi)v_{\text{triv}}, \quad \xi v_{\text{sgn}} = \rho(\xi)v_{\text{sgn}} \quad \text{for } \xi \in \mathfrak{a}_{\mathbb{C}},$$

then by (4.1) they become one-dimensional \mathbf{H} -modules, which are respectively called the *trivial* module and the *Steinberg* module. We denote them by $E_{\mathbf{H}}^1$ and $D_{\mathbf{H}}^1$.

Proposition 19.1. (i) *An irreducible \mathbf{H} -module is equivalent to exactly one of the following:*

$$(19.4) \quad \begin{cases} B_{\mathbf{H}}(\lambda) \simeq B_{\mathbf{H}}(-\lambda) \text{ with } \lambda \neq \pm \rho, \\ E_{\mathbf{H}}^1, \\ D_{\mathbf{H}}^1. \end{cases}$$

(ii) $B_{\mathbf{H}}(\pm \rho)$ are indecomposable and

$$\begin{aligned} E_{\mathbf{H}}^1 &\subset B_{\mathbf{H}}(\rho), & B_{\mathbf{H}}(\rho)/E_{\mathbf{H}}^1 &\simeq D_{\mathbf{H}}^1, \\ D_{\mathbf{H}}^1 &\subset B_{\mathbf{H}}(-\rho), & B_{\mathbf{H}}(-\rho)/D_{\mathbf{H}}^1 &\simeq E_{\mathbf{H}}^1. \end{aligned}$$

(iii)

$$X_{\mathbf{H}}(\lambda) \simeq \begin{cases} B_{\mathbf{H}}(\lambda) \simeq \mathcal{A}(A, \lambda) & \text{if } \lambda \neq \pm \rho, \\ E_{\mathbf{H}}^1 & \text{if } \lambda = \pm \rho. \end{cases}$$

(iv) For $\lambda \neq -\rho$

$$P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \lambda) \simeq B_{\mathbf{H}}(-\lambda).$$

Proof. The bijectivity condition for $\mathcal{P}_{\mathbf{H}}^\lambda$ in Proposition 12.1 reduces to

$$\lambda \neq -\rho$$

in the current case. Hence (iv) follows from Proposition 9.5 and Theorem 11.4 (ii).

Since the irreducibility condition (13.6) for $B_{\mathbf{H}}(\lambda)$ reduces to $\lambda \neq \pm \rho$, we have

$$(19.5) \quad B_{\mathbf{H}}(\lambda) \simeq \mathcal{A}(A, \lambda) = X_{\mathbf{H}}(\lambda) = \mathcal{A}(A, -\lambda) \simeq B_{\mathbf{H}}(-\lambda) \quad \text{for } \lambda \neq \pm \rho.$$

Observe $B_{\mathbf{H}}(\pm \lambda)$ has central character $[\lambda] = [-\lambda]$. Hence $B_{\mathbf{H}}(\lambda) \not\simeq B_{\mathbf{H}}(\mu)$ if $\lambda \neq \pm \mu$. Since $B_{\mathbf{H}}(\lambda) \simeq \mathbb{C}W$ as a W -module, one sees (19.4) is a list of inequivalent irreducible \mathbf{H} -modules. In order to show this list is complete, suppose \mathcal{X} is any irreducible \mathbf{H} -module. Then \mathcal{X} has a central character, say $[\lambda]$ ($\lambda \in \mathfrak{a}_{\mathbb{C}}^*$). Furthermore \mathcal{X} must contain an irreducible W -submodule F which is equivalent to \mathbb{C}_{triv} or \mathbb{C}_{sgn} . First, we assume $F \simeq \mathbb{C}_{\text{triv}}$. Then there exists a surjective \mathbf{H} -homomorphism

$$P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \lambda) = P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, -\lambda) \twoheadrightarrow \mathcal{X}.$$

If $\lambda \neq \pm\rho$ then by (iv) we have $\mathcal{X} \simeq B_{\mathbf{H}}(-\lambda)$, which is listed in (19.4). If $\lambda = \pm\rho$ then by (iv) again \mathcal{X} is a quotient of $B_{\mathbf{H}}(-\rho)$. On the other hand, since $\mathcal{P}_{\mathbf{H}}^{-\rho} \mathbf{1}_{\mathbf{H}}^{-\rho} = \gamma(\phi_{-\rho}) = \gamma(\phi_{\rho}) \in X_{\mathbf{H}}(\rho)$, we have the exact sequence

$$(19.6) \quad 0 \rightarrow \text{Ker } \mathcal{P}_{\mathbf{H}}^{-\rho} \rightarrow B_{\mathbf{H}}(-\rho) \rightarrow X_{\mathbf{H}}(\rho) \rightarrow 0$$

where $\text{Ker } \mathcal{P}_{\mathbf{H}}^{-\rho} \simeq \mathbb{C}_{\text{sgn}}$ and $X_{\mathbf{H}}(\rho) \simeq \mathbb{C}_{\text{triv}}$ as W -modules. This implies $\mathcal{X} \simeq X_{\mathbf{H}}(\rho)$. By considering the special case of $\mathcal{X} = E_{\mathbf{H}}^1$ we get

$$(19.7) \quad E_{\mathbf{H}}^1 \simeq X_{\mathbf{H}}(\rho).$$

Secondly, assume $F \simeq \mathbb{C}_{\text{sgn}}$. Then there exists a surjective \mathbf{H} -homomorphism

$$P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}, \lambda) = P_{\mathbf{H}}(\mathbb{C}_{\text{sgn}}, -\lambda) \twoheadrightarrow \mathcal{X}.$$

If $\lambda \neq \pm\rho$ then from (19.5) and Theorem 11.4 (v) we have $B_{\mathbf{H}}(\lambda) \simeq \mathcal{X}$. If $\lambda = \pm\rho$ then \mathcal{X} is a quotient of $\mathcal{A}(A, \rho)$ by Theorem 11.4 (v). Since $\mathcal{A}(A, \rho)$ contains $X_{\mathbf{H}}(\rho)$ as a unique irreducible subspace by Theorem 11.4 (iii) and $\dim X_{\mathbf{H}}(\rho) = 1$, we have $\mathcal{X} \simeq \mathcal{A}(A, \rho)/X_{\mathbf{H}}(\rho)$. By considering two special cases of $\mathcal{X} = D_{\mathbf{H}}^1$ and $\mathcal{X} = \text{Ker } \mathcal{P}_{\mathbf{H}}^{-\rho}$ we get

$$(19.8) \quad D_{\mathbf{H}}^1 \simeq \text{Ker } \mathcal{P}_{\mathbf{H}}^{-\rho} \simeq \mathcal{A}(A, \rho)/X_{\mathbf{H}}(\rho).$$

Hence in either case \mathcal{X} is equivalent to one of (19.4).

Now $B_{\mathbf{H}}(\rho) \simeq \mathcal{A}(A, \rho)$ is clearly indecomposable and we have from (19.7) and (19.8) the exact sequence

$$0 \rightarrow E_{\mathbf{H}}^1 \rightarrow B_{\mathbf{H}}(\rho) \rightarrow D_{\mathbf{H}}^1 \rightarrow 0.$$

Since $B_{\mathbf{H}}(-\rho) \simeq P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, \rho)$ is generated by the unique one-dimensional W -invariant subspace, $B_{\mathbf{H}}(-\rho)$ is also indecomposable. In addition, from (19.6), (19.7) and (19.8) we have the exact sequence

$$0 \rightarrow D_{\mathbf{H}}^1 \rightarrow B_{\mathbf{H}}(-\rho) \rightarrow E_{\mathbf{H}}^1 \rightarrow 0.$$

Thus (ii) is proved.

Finally (iii) follows from (19.5) and (19.7). \square

Since $G = SL(2, \mathbb{R})$ is split, $\mathfrak{b} = \{0\}$ and $\rho_{\mathfrak{m}} = 0$ in Theorem 16.6. Let $(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{\mathbf{H}}^{\text{fl}}$ be the full subcategory of $(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}^{\text{fl}}$ which consists of the $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules of finite length having all K -types in \widehat{K}_M . For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let $(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{M, [\lambda]}^{\text{fl}}$ denote the full subcategory of $(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{\mathbf{H}}^{\text{fl}}$ consisting of the objects with generalized infinitesimal character $[\lambda]$ and let $\mathbf{H}\text{-Mod}_{[\lambda]}^{\text{fd}}$ denote the full subcategory of $\mathbf{H}\text{-Mod}^{\text{fd}}$ consisting of the finite-dimensional \mathbf{H} -modules with generalized central character $[\lambda]$. Then we have

$$(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{\mathbf{H}}^{\text{fl}} = \bigoplus_{[\lambda]} (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{M, [\lambda]}^{\text{fl}}, \quad \mathbf{H}\text{-Mod}^{\text{fd}} = \bigoplus_{[\lambda]} \mathbf{H}\text{-Mod}_{[\lambda]}^{\text{fd}}$$

and it follows from Theorems 16.6 and 16.7 that for each $[\lambda]$ three functors Ξ_{rad} , Ξ^{min} and Ξ send an object in $\mathbf{H}\text{-Mod}_{[\lambda]}^{\text{fd}}$ into $(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{M, [\lambda]}^{\text{fl}}$.

Theorem 19.2. (i) On $\mathbf{H}\text{-Mod}^{\text{fd}}$ two functors Ξ_{rad} and Ξ^{\min} are exact and coincide with each other. For λ with $\lambda(\alpha^\vee) \notin \{\pm 3, \pm 5, \dots\}$, three functors Ξ_{rad} , Ξ^{\min} and Ξ coincide on $\mathbf{H}\text{-Mod}_{[\lambda]}^{\text{fd}}$ (and hence are exact there).

(ii)

$$\begin{aligned}\Xi_{\text{rad}}(B_{\mathbf{H}}(\lambda)) &= \Xi^{\min}(B_{\mathbf{H}}(\lambda)) = \Xi(B_{\mathbf{H}}(\lambda)) = B_G(\lambda)_{K\text{-finite}} \quad \text{if } \lambda(\alpha^\vee) \notin \{\pm 3, \pm 5, \dots\}, \\ \Xi_{\text{rad}}(E_{\mathbf{H}}^1) &= \Xi^{\min}(E_{\mathbf{H}}^1) = \Xi(E_{\mathbf{H}}^1) = E_G^1, \\ \Xi_{\text{rad}}(D_{\mathbf{H}}^1) &= \Xi^{\min}(D_{\mathbf{H}}^1) = \Xi(D_{\mathbf{H}}^1) = D_G^{1,+} \oplus D_G^{1,-}.\end{aligned}$$

(iii) For $n = 3, 5, \dots$

$$\begin{aligned}\Xi_{\text{rad}}(B_{\mathbf{H}}(-n\rho)) &= \Xi^{\min}(B_{\mathbf{H}}(-n\rho)) = B_G(-n\rho)_{K\text{-finite}}, \\ \Xi(B_{\mathbf{H}}(-n\rho)) &= E_G^n.\end{aligned}$$

(Note $B_{\mathbf{H}}(n\rho) = B_{\mathbf{H}}(-n\rho)$ in this case.)

(iv) For $n = 3, 5, \dots$ the functor Ξ is not exact on $\mathbf{H}\text{-Mod}_{[n\rho]}^{\text{fd}}$.

Proof. Suppose $\lambda(\alpha^\vee) \notin \{\pm 3, \pm 5, \dots\}$. Then each irreducible object in $(\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{M, [\lambda]}^{\text{fl}}$ contains at least one single-petaled K -type and conversely any single-petaled K -type is contained in exactly one irreducible object with multiplicity 1:

$$\begin{aligned}\lambda(\alpha^\vee) \notin \{\pm 1, \pm 3, \dots\} : & B_G(\lambda)_{K\text{-finite}} \supset \mathbb{C}_{\text{triv}}, \mathfrak{s}_{\pm}, \\ \lambda = \pm \rho : & E_G \supset \mathbb{C}_{\text{triv}}, D_G^{1,+} \supset \mathfrak{s}_+, D_G^{1,-} \supset \mathfrak{s}_-.\end{aligned}$$

Now for $\mathcal{X} \in \mathbf{H}\text{-Mod}_{[\lambda]}^{\text{fd}}$ we have consecutive surjective $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphisms

$$(19.9) \quad \Xi_{\text{rad}}(\mathcal{X}) \twoheadrightarrow \Xi^{\min}(\mathcal{X}) \twoheadrightarrow \Xi(\mathcal{X}).$$

They are in fact natural transforms. We assert if $\mathcal{J} \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{M, [\lambda]}^{\text{fl}}$ is an irreducible object then it appears with the same multiplicity in the three composition series for $\Xi_{\text{rad}}(\mathcal{X})$, $\Xi^{\min}(\mathcal{X})$ and $\Xi(\mathcal{X})$. Indeed, if \mathcal{J} contains $V \in \widehat{K}_{\text{sp}}$ then we have from (8.11)

$$\text{Hom}_K(V, \Xi_{\text{rad}}(\mathcal{X})) \simeq \text{Hom}_K(V, \Xi^{\min}(\mathcal{X})) \simeq \text{Hom}_K(V, \Xi(\mathcal{X})) \simeq \text{Hom}_W(V^M, \mathcal{X}).$$

Hence the multiplicity of \mathcal{J} in each series equals $\dim \text{Hom}_W(V^M, \mathcal{X})$. Thus two homomorphisms in (19.9) are bijective and $\Xi_{\text{rad}} = \Xi^{\min} = \Xi$ on $\mathbf{H}\text{-Mod}_{[\lambda]}^{\text{fd}}$. The exactness of these functors follows from Proposition 16.8 and Corollary 17.5.

If $\lambda(\alpha^\vee) \notin \{1, 3, \dots\}$ then by Proposition 19.1 (iv), (17.4) and (19.3)

$$(19.10) \quad \Xi^{\min}(B_{\mathbf{H}}(\lambda)) = \Xi^{\min}(P_{\mathbf{H}}(\mathbb{C}_{\text{triv}}, -\lambda)) = P_G(\mathbb{C}_{\text{triv}}, -\lambda) = B_G(\lambda)_{K\text{-finite}}.$$

Using this result for $\lambda = -\rho$, Proposition 9.5, Theorem 18.6 and Proposition 9.2, we obtain

$$\Xi(B_{\mathbf{H}}(\rho)) = \Xi(B_{\mathbf{H}}(-\rho)^*) = \Xi(B_{\mathbf{H}}(-\rho))^* = (B_G(-\rho)_{K\text{-finite}})^* = B_G(\rho)_{K\text{-finite}}.$$

Thus the first assertion of (ii) is proved.

Now it follows from Proposition 19.1 (iii), Corollary 18.5 (iii) and (19.2) that

$$\begin{aligned}\Xi(E_{\mathbf{H}}^1) &= \Xi(X_{\mathbf{H}}(\rho)) = X_G(\rho) = E_G^1, \\ \Xi(B_{\mathbf{H}}(-n\rho)) &= \Xi(X_{\mathbf{H}}(-n\rho)) = X_G(-n\rho) = E_G^n \quad \text{for } n = 3, 5, \dots\end{aligned}$$

This proves the second assertion of (ii) and the second assertion of (iii). The third assertion of (ii) follows from (19.1), Proposition 19.1 (ii) and the exactness of the three functors on $\mathbf{H}\text{-Mod}_{[\rho]}^{\text{fd}}$. Because of (19.10) the first assertion of (iii) follows if we can show the coincidence of Ξ_{rad} and Ξ^{\min} on $\mathbf{H}\text{-Mod}_{[n\rho]}^{\text{fd}}$ for $n = 3, 5, \dots$.

Suppose $n = 3, 5, \dots$. Let $C = (e_0^2 + 2e_+e_- + 2e_-e_+)/8 \in U(\mathfrak{g}_{\mathbb{C}})^G$ be the Casimir element. Then $\gamma(C) = ((\alpha^\vee)^2 - 1)/8$ and $\mathbb{C}[\gamma(C)] = S(\mathfrak{a}_{\mathbb{C}})^W$. Hence $\mathbb{C}[\gamma(C)] \otimes \mathbb{C}W$ is a subalgebra of $\mathbf{H} = S(\mathfrak{a}_{\mathbb{C}}) \otimes \mathbb{C}W = \mathbb{C}[\alpha^\vee] \otimes \mathbb{C}W$. Now, for any $\mathcal{X} \in \mathbf{H}\text{-Mod}_{[n\rho]}^{\text{fd}}$ the subspace \mathcal{X}^W of W -fixed elements is stable under the action of W and $\gamma(C)$. Hence an \mathbf{H} -homomorphism

$$(19.11) \quad \mathbf{H} \otimes_{\mathbb{C}[\gamma(C)] \otimes \mathbb{C}W} \mathcal{X}^W \longrightarrow \mathcal{X}.$$

is naturally defined. From the decomposition

$$\mathbf{H} = (\mathbb{C} \oplus \mathbb{C}(\alpha^\vee + 1)) \otimes \mathbb{C}[\gamma(C)] \otimes \mathbb{C}W$$

we have

$$(19.12) \quad \mathbf{H} \otimes_{\mathbb{C}[\gamma(C)] \otimes \mathbb{C}W} \mathcal{X}^W = 1 \otimes \mathcal{X}^W \oplus (\alpha^\vee + 1) \otimes \mathcal{X}^W.$$

It is easy to check by (4.1) this is the decomposition into the \mathbb{C}_{triv} - and \mathbb{C}_{sgn} -isotypic components as a W -module. Note also that $\mathbf{H} \otimes_{\mathbb{C}[\gamma(C)] \otimes \mathbb{C}W} \mathcal{X}^W \in \mathbf{H}\text{-Mod}_{[n\rho]}^{\text{fd}}$. We assert (19.11) is bijective. In fact, since the first summand of (19.12) is bijectively mapped to \mathcal{X}^W by (19.11), the kernel and the cokernel of (19.11) do not have any non-zero W -fixed vector. But since the unique irreducible object $B_{\mathbf{H}}(-n\rho)$ in $\mathbf{H}\text{-Mod}_{[n\rho]}^{\text{fd}}$ has a non-zero W -fixed vector, both the kernel and the cokernel must be zero. Now by [KR] the following decomposition holds:

$$(19.13) \quad U(\mathfrak{g}_{\mathbb{C}}) = (\mathbb{C} \oplus \bigoplus_{\nu \geq 1} (\mathbb{C}e_+^\nu \oplus \mathbb{C}e_-^\nu)) \otimes \mathbb{C}[C] \otimes U(\mathfrak{k}_{\mathbb{C}}).$$

In particular, $\mathbb{C}[C] \otimes U(\mathfrak{k}_{\mathbb{C}})$ is a subalgebra of $U(\mathfrak{g}_{\mathbb{C}})$. Regarding \mathcal{X}^W as a $(\mathbb{C}[C] \otimes U(\mathfrak{k}_{\mathbb{C}}), K)$ -module by the trivial K -action and the $\mathbb{C}[C]$ -action given by $Cx = \gamma(C)x$, we define the induced $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules

$$\tilde{\mathcal{Y}} := U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathcal{X}^W, \quad \mathcal{Y} := U(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathbb{C}[C] \otimes U(\mathfrak{k}_{\mathbb{C}})} \mathcal{X}^W.$$

If we put

$$\kappa : \tilde{\mathcal{Y}} \ni D \otimes x \longmapsto DC \otimes x - D \otimes \gamma(C)x \in \tilde{\mathcal{Y}}$$

then we have the exact sequence

$$\tilde{\mathcal{Y}} \xrightarrow{\kappa} \tilde{\mathcal{Y}} \rightarrow \mathcal{Y} \rightarrow 0.$$

Applying the right exact functor Γ in §6 to this, we get

$$\Gamma(\tilde{\mathcal{Y}}) \xrightarrow{\Gamma(\kappa)} \Gamma(\tilde{\mathcal{Y}}) \rightarrow \Gamma(\mathcal{Y}) \rightarrow 0, \quad (\text{exact})$$

$$\Gamma(\tilde{\mathcal{Y}}) = \mathbf{H} \otimes_{\mathbb{C}W} \mathcal{X}^W,$$

$$\Gamma(\kappa) : \mathbf{H} \otimes_{\mathbb{C}W} \mathcal{X}^W \ni h \otimes x \longmapsto h\gamma(C) \otimes x - h \otimes \gamma(C)x \in \mathbf{H} \otimes_{\mathbb{C}W} \mathcal{X}^W,$$

$$\therefore \Gamma(\mathcal{Y}) = \mathbf{H} \otimes_{\mathbb{C}[\gamma(C)] \otimes \mathbb{C}W} \mathcal{X}^W = \mathcal{X}.$$

Note the linear map $\gamma^{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{X}$ reduces to $D \otimes x \mapsto \gamma(D) \otimes x$. Let us prove $(\mathcal{Y}, \mathcal{X})$ is a radial pair and $\gamma^{\mathcal{Y}}$ is its radial restriction satisfying (rest-3). First, by 19.13 we can decompose \mathcal{Y} into the K -isotypic components:

$$\mathcal{Y} = (1 \otimes \mathcal{X}^W) \oplus (e_+ \otimes \mathcal{X}^W) \oplus (e_- \otimes \mathcal{X}^W) \oplus \bigoplus_{\nu \geq 2} ((e_+^\nu \otimes \mathcal{X}^W) \oplus (e_-^\nu \otimes \mathcal{X}^W)).$$

Thus all K -types of \mathcal{Y} belong to \widehat{K}_M . The first three summands in the decomposition correspond to \mathbb{C}_{triv} , \mathfrak{s}_+ and \mathfrak{s}_- , respectively. Take $m \in \mathbb{Z}_{\geq 0}$ so that $(\gamma(C) - \gamma(C)(n\rho))^m \mathcal{X}^W = \{0\}$. Suppose $\nu \in \mathbb{Z}_{\geq 0}$. Then one can directly calculate

$$\gamma(e_\pm^\nu) = \frac{1}{2^\nu}(\alpha^\vee + 1)(\alpha^\vee + 3) \cdots (\alpha^\vee + 2\nu - 1).$$

Hence there exists some $f_\nu \in S(\mathfrak{a}_{\mathbb{C}}) = \mathbb{C}[\alpha^\vee]$ such that

$$f_\nu \gamma(e_\pm^\nu) - (\alpha^\vee + n) \in S(\mathfrak{a}_{\mathbb{C}})(\gamma(C) - \gamma(C)(n\rho))^m.$$

The restriction of $\gamma^{\mathcal{Y}}$ to each K -isotypic component $e_\pm^\nu \otimes \mathcal{X}^W$ is injective since the composition of this map with multiplication by f_ν reduces to

$$e_\pm^\nu \otimes \mathcal{X}^W \ni e_\pm^\nu \otimes x \xrightarrow{\gamma^{\mathcal{Y}}} \gamma(e_\pm^\nu) \otimes x \xrightarrow{f_\nu} f_\nu \gamma(e_\pm^\nu) \otimes x = (\alpha^\vee + n) \otimes x \in \mathcal{X}.$$

If $\nu = 0, \pm 1$ we also have

$$\gamma^{\mathcal{Y}}(1 \otimes \mathcal{X}^W) = 1 \otimes \mathcal{X}^W, \quad \gamma^{\mathcal{Y}}(e_+ \otimes \mathcal{X}^W) = \gamma^{\mathcal{Y}}(e_- \otimes \mathcal{X}^W) = (\alpha^\vee + 1) \otimes \mathcal{X}^W.$$

Thus $\gamma^{\mathcal{Y}}$ is a radial restriction defining a structure of $(\mathcal{Y}, \mathcal{X})$ as an object of \mathcal{C}_{Ch} . Since $\gamma^{\mathcal{Y}}$ satisfies (rest-3) by Remark 8.14, Lemma 8.13 implies $(\mathcal{Y}, \mathcal{X}) \in \mathcal{C}_{\text{rad}}$. Now $\Xi_{((\mathcal{Y}, \mathcal{X}))}^{\min}(\mathcal{X}) = \mathcal{Y}$ since \mathcal{Y} is generated by $1 \otimes \mathcal{X}^W$. Hence by Theorem 17.3 we have $\Xi^{\min}(\mathcal{X}) = \mathcal{Y}$ and the functor Ξ^{\min} restricted to $\mathbf{H}\text{-Mod}_{[n\rho]}^{\text{fd}}$ coincides with the functor

$$\mathcal{X} \longmapsto U(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathbb{C}[C] \otimes U(\mathfrak{k}_{\mathbb{C}})} \mathcal{X}^W.$$

This is exact by (19.13). We must prove this is also equal to Ξ_{rad} . To do so recall the linear map $\gamma_{\text{rad}} : \Xi_{\text{rad}}(\mathcal{X}) \rightarrow \mathcal{X}$ used in §16 satisfies Conditions (rest-2) and (rest-3). Thus γ_{rad} induces the linear bijection

$$\gamma_{\text{rad}} : \Xi_{\text{rad}}(\mathcal{X})^K \xrightarrow{\sim} \mathcal{X}^W$$

by (rest-2) and satisfies

$$\gamma_{\text{rad}}(Cy) = \gamma(C)\gamma_{\text{rad}}(y) \quad \text{for } y \in \Xi_{\text{rad}}(\mathcal{X})^K$$

by (rest-3). Hence we can define a $(\mathfrak{g}_{\mathbb{C}}, K)$ -homomorphism $\mathcal{I} : U(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathbb{C}[C] \otimes U(\mathfrak{k}_{\mathbb{C}})} \mathcal{X}^W \rightarrow \Xi_{\text{rad}}(\mathcal{X})$ by

$$D \otimes x \longmapsto Dy \text{ with } y \in \Xi_{\text{rad}}(\mathcal{X})^K \text{ such that } \gamma_{\text{rad}}(y) = x.$$

If $\mathcal{J} \in (\mathfrak{g}_{\mathbb{C}}, K)\text{-Mod}_{M, [n\rho]}^{\text{fl}}$ is a composition factor of $\text{Coker } \mathcal{I}$, then it does not contain the trivial K -type. This means $\mathcal{J} = D_G^{n,+}$ or $D_G^{n,-}$. But since each of $D_G^{n,\pm}$ contains neither \mathfrak{s}_+ nor \mathfrak{s}_- and since $\Xi_{\text{rad}}(\mathcal{X})$ is generated by the sum of those isotypic components for \mathbb{C}_{triv} , \mathfrak{s}_+ and \mathfrak{s}_- , we conclude $\text{Coker } \mathcal{I} = \{0\}$ and \mathcal{I} is surjective. Hence the natural

surjective homomorphism $\Xi_{\text{rad}}(\mathcal{X}) \rightarrow \Xi^{\min}(\mathcal{X})$ must be a bijection, proving $\Xi_{\text{rad}} = \Xi^{\min}$ on $\mathbf{H}\text{-Mod}_{[n\rho]}^{\text{fd}}$.

Finally in order to prove (iv), suppose $n = 3, 5, \dots$ as in the last paragraph and put

$$\mathcal{U}_j = \mathbb{C}[\gamma(C)] / \mathbb{C}[\gamma(C)](\gamma(C) - \gamma(C)(n\rho))^j \quad (j = 1, 2).$$

This is a $\mathbb{C}[\gamma(C)] \otimes \mathbb{C}W$ -module by the trivial W -action and is also a $(\mathbb{C}[C] \otimes U(\mathfrak{k}_{\mathbb{C}}), K)$ -module by the trivial K -action and the $\mathbb{C}[C]$ -action given by $Cu = \gamma(C)u$. Put

$$\mathcal{X}_j = \mathbf{H} \otimes_{\mathbb{C}[\gamma(C)] \otimes \mathbb{C}W} \mathcal{U}_j, \quad \mathcal{Y}_j = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathbb{C}[C] \otimes U(\mathfrak{k}_{\mathbb{C}})} \mathcal{U}_j \quad (j = 1, 2).$$

Then by the above argument we have $\Xi_{\text{rad}}(\mathcal{X}_j) = \mathcal{Y}_j$ and the exact sequence

$$0 \longrightarrow \mathcal{U}_1 \xrightarrow{\text{multiplication by } \gamma(C) - \gamma(C)(n\rho)} \mathcal{U}_2 \xrightarrow{\text{quotient map}} \mathcal{U}_1 \longrightarrow 0$$

induces exact sequences

$$0 \rightarrow \mathcal{X}_1 \rightarrow \mathcal{X}_2 \rightarrow \mathcal{X}_1 \rightarrow 0, \quad 0 \rightarrow \mathcal{Y}_1 \rightarrow \mathcal{Y}_2 \rightarrow \mathcal{Y}_1 \rightarrow 0.$$

We shall prove

$$(19.14) \quad 0 \rightarrow \Xi(\mathcal{X}_1) \rightarrow \Xi(\mathcal{X}_2) \rightarrow \Xi(\mathcal{X}_1) \rightarrow 0$$

is not exact. Since $\dim \mathcal{X}_1 = 2$, we see $\mathcal{X}_1 = B_{\mathbf{H}}(-n\rho)$ by Proposition 19.1 (i). Hence $\Xi(\mathcal{X}_1) = E_G^n$ by (iii) of the theorem. Since E_G^n does not contain the K -type corresponding to $n+1$, it suffices to show $\Xi(\mathcal{X}_2)$ has this K -type. By definition we have $\Xi(\mathcal{X}_2) = \mathcal{Y}_2 / \mathcal{N}$ with

$$\mathcal{N} = \sum \{ \mathcal{Y} \subset \mathcal{Y}_2; \text{ a } (\mathfrak{g}_{\mathbb{C}}, K)\text{-submodule containing no single-petaled } K\text{-type} \}.$$

We assert $e_+^{\frac{n+1}{2}} \otimes 1 \in \mathcal{Y}_2$ does not belong to \mathcal{N} . Indeed, if it does, then since

$$e_- e_+^{\frac{n+1}{2}} = e_+^{\frac{n-1}{2}} \left(2(C - \gamma(C)(n\rho)) - \frac{n}{2}e_0 - \frac{1}{4}e_0^2 \right) \quad \text{in } U(\mathfrak{g}_{\mathbb{C}}),$$

\mathcal{N} must contain

$$\mathbb{C}e_-(e_+^{\frac{n+1}{2}} \otimes 1) = \mathbb{C}e_+^{\frac{n-1}{2}} \otimes 2(\gamma(C) - \gamma(C)(n\rho)),$$

a K -type corresponding to $n-1$; Hence in the composition series of \mathcal{N} there appears the irreducible object E_G^n , whose K -types are

$$n-1, n-3, \dots, 2, 0, -2, \dots, -n+3, -n+1.$$

This contradicts the definition of \mathcal{N} . Thus $\Xi(\mathcal{X}_2)$ contains

$$\mathbb{C}(e_+^{\frac{n+1}{2}} \otimes 1 \bmod \mathcal{N}),$$

a K -type corresponding to $n+1$. Hence (19.14) is not exact. \square

APPENDIX A. NON-SYMMETRIC HYPERGEOMETRIC FUNCTIONS

Let $\mathbf{k} : W \setminus R \rightarrow \mathbb{C}$ and $\mathcal{T}_{\mathbf{k}}$ be as in Definition 4.2. Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Opdam's non-symmetric hypergeometric function $\mathbf{G}(\lambda, \mathbf{k}, a)$ is an analytic function on A satisfying

$$\begin{cases} \mathcal{T}_{\mathbf{k}}(\xi)\mathbf{G}(\lambda, \mathbf{k}, a) = \lambda(\xi)\mathbf{G}(\lambda, \mathbf{k}, a) & \text{for } \xi \in \mathfrak{a}_{\mathbb{C}}, \\ \mathbf{G}(\lambda, \mathbf{k}, 1) = 1. \end{cases}$$

Opdam shows in [Op1, §3] that there uniquely exists such a function for a generic \mathbf{k} . This is the case when $\mathbf{k} = \mathbf{m}$ and $\mathbf{G}(\lambda, a) := \mathbf{G}(\lambda, \mathbf{m}, a)$ plays central roles in §§11–14. The purpose of the appendix is to prove the following:

Theorem A.1. *Suppose $\varphi(a) \in C^\infty(A)$ satisfies*

$$(A.1) \quad \mathcal{T}_{\mathbf{k}}(\xi)\varphi(a) = \lambda(\xi)\varphi(a) \quad \text{for } \xi \in \mathfrak{a}_{\mathbb{C}}.$$

Then $\varphi(a) \in \mathcal{A}(A)$.

Recall Lemma 11.3 is founded on this result. Let us start with an elementary lemma.

Lemma A.2. *Suppose $\epsilon > 0$ and a C^∞ function $F(x)$ on $(-\epsilon, \epsilon)$ satisfies the following conditions:*

- (i) $F(x)$ is analytic on $(-\epsilon, 0) \sqcup (0, \epsilon)$;
- (ii) there exist $f_{\lambda,j} \in \mathcal{A}((-\epsilon, \epsilon))$ with a finite index set $\Lambda \subset \mathbb{C} \times \mathbb{Z}_{\geq 0}$ such that

$$F(x) = \sum_{(\lambda,j) \in \Lambda} x^\lambda (\log x)^j f_{\lambda,j}(x) \quad \text{on } (0, \epsilon);$$

- (iii) there exist $g_{\lambda,j} \in \mathcal{A}((-\epsilon, \epsilon))$ with a finite index set $\Lambda' \subset \mathbb{C} \times \mathbb{Z}_{\geq 0}$ such that

$$F(x) = \sum_{(\lambda,j) \in \Lambda'} (-x)^\lambda (\log(-x))^j g_{\lambda,j}(x) \quad \text{on } (-\epsilon, 0)$$

Then $F(x)$ is analytic on $(-\epsilon, \epsilon)$.

Proof. Without loss of generality we may assume in Condition (ii)

$$f_{\lambda,j}(0) \neq 0 \quad \text{for all } (\lambda, j) \in \Lambda.$$

We then assert $\lambda \in \mathbb{Z}_{\geq 0}$ for any $(\lambda, j) \in \Lambda$. To prove this, suppose $\lambda \notin \mathbb{Z}_{\geq 0}$ for some $(\lambda, j) \in \Lambda$. Then the derivative $F^{(k)}$ of F with a sufficiently high order k has an expression

$$F^{(k)}(x) = \sum_{(\lambda,j) \in \tilde{\Lambda}} x^\lambda (\log x)^j h_{\lambda,j}(x) \quad \text{on } (0, \epsilon)$$

with $h_{\lambda,j}(0) \neq 0$ for all $(\lambda, j) \in \tilde{\Lambda}$ and $\operatorname{Re} \lambda < 0$ for some $(\lambda, j) \in \tilde{\Lambda}$. Putting

$$\lambda^\circ = \min\{\operatorname{Re} \lambda; (\lambda, j) \in \tilde{\Lambda}\},$$

$$j^\circ = \max\{j; (\lambda, j) \in \tilde{\Lambda} \text{ with } \operatorname{Re} \lambda = \lambda^\circ\},$$

$$\tilde{\Lambda}^\circ = \{(\lambda, j) \in \tilde{\Lambda}; \operatorname{Re} \lambda = \lambda^\circ, j = j^\circ\},$$

we have for $x \in (0, \min\{\epsilon, 1\})$

$$|F^{(k)}(x)| = x^{\lambda^\circ} |\log x|^{j^\circ} \left| \sum_{(\lambda,j) \in \tilde{\Lambda}^\circ} x^{\lambda-\lambda^\circ} h_{\lambda,j}(0) + \sum_{(\lambda,j) \in \tilde{\Lambda}^\circ} x^{\lambda-\lambda^\circ+1} \frac{h_{\lambda,j}(x) - h_{\lambda,j}(0)}{x} \right. \\ \left. + \sum_{(\lambda,j) \in \tilde{\Lambda} \setminus \tilde{\Lambda}^\circ} x^{\lambda-\lambda^\circ} (\log x)^{j-j^\circ} h_{\lambda,j}(x) \right|$$

But since

$$\limsup_{x \downarrow 0} \left| \sum_{(\lambda,j) \in \tilde{\Lambda}^\circ} x^{\lambda-\lambda^\circ} h_{\lambda,j}(0) \right| \geq \left(\sum_{(\lambda,j) \in \tilde{\Lambda}^\circ} |h_{\lambda,j}(0)|^2 \right)^{1/2}, \quad (\because [\text{Hel4, Ch. I, Exercise D5}]) \\ \lim_{x \downarrow 0} \left| \sum_{(\lambda,j) \in \tilde{\Lambda}^\circ} x^{\lambda-\lambda^\circ+1} \frac{h_{\lambda,j}(x) - h_{\lambda,j}(0)}{x} \right| = \lim_{x \downarrow 0} \left| \sum_{(\lambda,j) \in \tilde{\Lambda} \setminus \tilde{\Lambda}^\circ} x^{\lambda-\lambda^\circ} (\log x)^{j-j^\circ} h_{\lambda,j}(x) \right| = 0,$$

$\limsup_{x \downarrow 0} |F^{(k)}(x)| = \infty$, contradicting the fact $F(x) \in C^\infty((-\epsilon, \epsilon))$. Thus our assertion is proved and we may assume F has an expression

$$F(x) = \sum_{j \in J} x^{n_j} (\log x)^j f_j(x) \quad \text{on } (0, \epsilon)$$

where $J \subset \mathbb{Z}_{\geq 0}$ is a finite set and for each $j \in J$

$$n_j \in \mathbb{Z}_{\geq 0}, \quad f_j \text{ is analytic on } (-\epsilon, \epsilon) \text{ with } f_j(0) \neq 0.$$

Now let us assume J contains some $j > 0$ and lead a contradiction. Put

$$n' = \min\{n_j; j \in J \setminus \{0\}\}, \quad j' = \max\{j; j \in J \setminus \{0\} \text{ with } n_j = n'\}.$$

Then there exist $h_j \in \mathcal{A}((-\epsilon, \epsilon))$ ($j = 0, 1, \dots, \max J$) such that

$$F^{(n'+1)}(x) = \sum_j x^{-1} (\log x)^j h_j(x), \quad h_{j'-1}(0) \neq 0, \quad h_j(0) = 0 \text{ for } j \geq j'.$$

This implies

$$\lim_{x \downarrow 0} \frac{F^{(n'+1)}(x)}{x^{-1} (\log x)^{j'-1}} = h_{j'-1}(0) \neq 0,$$

a contradiction. Hence $J = \emptyset$ or $\{0\}$ and $F|_{(0,\epsilon)}$ extends to an analytic function F_1 on $(-\epsilon, \epsilon)$. Similarly one can prove $F|_{(-\epsilon,0)}$ extends to an analytic function F_2 on $(-\epsilon, \epsilon)$. Since $F_1^{(k)}(0) = F_2^{(k)}(0) = F^{(k)}(0)$ for any k , we conclude $F_1 = F_2 = F$. \square

Put

$$\begin{aligned} \mathfrak{a}_{\text{reg}} &= \{H \in \mathfrak{a}; \alpha(H) \neq 0 \text{ for any } \alpha \in R\}, \\ \mathfrak{a}_+ &= \{H \in \mathfrak{a}; \alpha(H) > 0 \text{ for any } \alpha \in \Pi\}, \\ U &= \{H \in \mathfrak{a}; |\alpha(H)| < 2\pi \text{ for any } \alpha \in R\}, \\ U_+ &= U \cap \mathfrak{a}_+, \\ (\mathfrak{a} + iU)_{\text{reg}} &= \{H \in \mathfrak{a} + iU; \alpha(H) \neq 0 \text{ for any } \alpha \in R\}. \end{aligned}$$

Definition A.3 (the Knizhnik-Zamolodchikov connection [M]). Let $E = (\mathfrak{a} + iU) \times \mathbb{C}W$ be the trivial vector bundle with fiber $\mathbb{C}W$ over the complex manifold $\mathfrak{a} + iU$. The *Knizhnik-Zamolodchikov connection* $\nabla = \nabla(\lambda, \mathbf{k})$ is a connection on $(\mathfrak{a} + iU)_{\text{reg}} \times \mathbb{C}W \subset E$ whose covariant derivative along $\xi \in \mathfrak{a}_{\mathbb{C}}$ is given by

$$(A.2) \quad \nabla_{\xi} \Psi = \sum_{w \in W} \left(\left(\partial(\xi) - \lambda(w^{-1}\xi) + \frac{1}{2} \sum_{\alpha \in R^+} \mathbf{k}(\alpha) \alpha(\xi) \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \right) \Psi_w - \sum_{\alpha \in wR^+} \mathbf{k}(\alpha) \alpha(\xi) \frac{e^{-\alpha}}{1 - e^{-\alpha}} \Psi_{s_{\alpha}w} \right) w$$

for any section $\Psi : H \mapsto \Psi(H) = \sum_{w \in W} \Psi_w(H)w \in \mathbb{C}W$.

Remark A.4. It is known that the KZ connection is integrable (cf. [M, Proposition 3.3.1]), although we do not need this fact here.

Now suppose $\varphi(a) \in C^{\infty}(A)$ satisfies (A.1) and define a $\mathbb{C}W$ -valued C^{∞} function

$$\Phi(H) = \sum_{w \in W} \varphi(\exp(w^{-1}H))w$$

on \mathfrak{a} . Then it follows from [Op1, Lemma 3.2] that

$$(A.3) \quad \nabla_{\xi} \Phi(H) = 0 \quad \text{for any } \xi \in \mathfrak{a} \text{ and } H \in \mathfrak{a}_{\text{reg}}.$$

Hence it suffices to deduce from (A.3) the analyticity of Φ on \mathfrak{a} . If $\{\xi_1, \dots, \xi_{\ell}\}$ is a basis of \mathfrak{a} , then the system $\nabla_{\xi} \Psi = 0$ ($\xi \in \mathfrak{a}$) for a holomorphic section Ψ of $(\mathfrak{a} + iU)_{\text{reg}} \times \mathbb{C}W \subset E$ can be written as

$$(A.4) \quad \partial(\xi_j) \Psi = A_j \Psi \quad (j = 1, \dots, \ell)$$

where $A_j : (\mathfrak{a} + iU)_{\text{reg}} \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}W)$ are holomorphic functions. Hence Φ is analytic on $\mathfrak{a}_{\text{reg}} = \bigsqcup_{w \in W} w\mathfrak{a}_+$ and $\Phi|_{\mathfrak{a}_+}$ extends to a global (possibly multi-valued) holomorphic solution $\tilde{\Phi}$ of (A.4) on the whole $(\mathfrak{a} + iU)_{\text{reg}}$ (cf. [Kn, Appendix B, §2]).

Lemma A.5. *The global solution $\tilde{\Phi}$ is single-valued on $(\mathfrak{a} + iU)_{\text{reg}}$ and $\tilde{\Phi}|_{\mathfrak{a}_{\text{reg}}} = \Phi|_{\mathfrak{a}_{\text{reg}}}$.*

Proof. Note

$$(\mathfrak{a} + iU)_{\text{reg}} = \bigsqcup_{w \in W} (w\mathfrak{a}_+ + iU) \cup \bigsqcup_{t \in W} (\mathfrak{a} + itU_+)$$

and for any $w, t \in W$

$$w\mathfrak{a}_+ + iU, \quad \mathfrak{a} + itU_+, \quad (w\mathfrak{a}_+ + iU) \cap (\mathfrak{a} + itU_+) = w\mathfrak{a}_+ + itU_+$$

are all simply connected. Hence it suffices to prove that for any $w_1, w_2, t \in W$

$$\Phi|_{w_1\mathfrak{a}_+} \xrightarrow{\text{extension}} \Phi_1 \text{ on } w_1\mathfrak{a}_+ + iU \xrightarrow{\text{restriction}} \Phi_1|_{w_1\mathfrak{a}_+ + itU_+} \xrightarrow{\text{extension}} \tilde{\Phi}_1 \text{ on } \mathfrak{a} + itU_+$$

and

$$\Phi|_{w_2\mathfrak{a}_+} \xrightarrow{\text{extension}} \Phi_2 \text{ on } w_2\mathfrak{a}_+ + iU \xrightarrow{\text{restriction}} \Phi_2|_{w_2\mathfrak{a}_+ + itU_+} \xrightarrow{\text{extension}} \tilde{\Phi}_2 \text{ on } \mathfrak{a} + itU_+$$

give the same result. Clearly we may assume $w_2 = w_1 s_\alpha$ for some $\alpha \in \Pi$. Fix an arbitrary $\xi_0 \in \mathfrak{a}_+$ and put $H_0 = w_1 \xi_0 + w_2 \xi_0$. Then

$$(w_1 \alpha)(H_0) = (w_2 \alpha)(H_0) = 0, \quad (w_1 \beta)(H_0), (w_2 \beta)(H_0) > 0 \text{ for any } \beta \in R_1^+ \setminus \{\alpha\}.$$

Take a sufficiently small $\epsilon > 0$ so that

$$\{H_0 + zt\xi_0; z \in \mathbb{C} \text{ with } 0 < |z| < \epsilon\} \subset (\mathfrak{a} + iU)_{\text{reg}}.$$

Then from (A.2) one sees the covariant derivative $\nabla_{t\xi_0}$ on $\{H = H_0 + zt\xi_0; 0 < |z| < \epsilon\}$ is written as

$$\nabla_{t\xi_0} \Psi = \left(\frac{d}{dz} - \frac{B(z)}{z} \right) \Psi$$

where B is an $\text{End}_{\mathbb{C}}(\mathbb{C}W)$ -valued holomorphic function on $\{z \in \mathbb{C}; |z| < \epsilon\}$. Now let us consider the $\mathbb{C}W$ -valued C^∞ function

$$\Psi(x) = \sum_{w \in W} \Psi_w(x) w := \Phi(H_0 + xt\xi_0)$$

on $(-\epsilon, \epsilon)$. Since Ψ is a solution of the first-order ordinary linear system

$$\frac{d}{dx} \Psi = \frac{B(x)}{x} \Psi$$

with a regular singular point $x = 0$, each Ψ_w ($w \in W$) satisfies the assumption of Lemma A.2. Hence Ψ is analytic at $x = 0$ and extends to a holomorphic function $\tilde{\Psi}$ on $\{z \in \mathbb{C}; |z| < \epsilon\}$. Now suppose $(w_1 \alpha)(t\xi_0) > 0$. (If $(w_1 \alpha)(t\xi_0) < 0$ we swap w_1 and w_2 .) Identifying $\{z \in \mathbb{C}; |z| < \epsilon\}$ with a subset of $\mathfrak{a} + iU$ by $H = H_0 + zt\xi_0$ we have

$$\begin{aligned} D_+ &:= \{z \in \mathbb{C}; |z| < \epsilon, \text{Im } z > 0\} \subset \mathfrak{a} + itU_+, \\ &\{z \in \mathbb{C}; |z| < \epsilon, \text{Re } z > 0\} \subset w_1 \mathfrak{a}_+ + iU, \\ &\{z \in \mathbb{C}; |z| < \epsilon, \text{Re } z < 0\} \subset w_2 \mathfrak{a}_+ + iU. \end{aligned}$$

Thus $\tilde{\Phi}_1|_{D_+} = \tilde{\Phi}_2|_{D_+} = \tilde{\Psi}|_{D_+}$. Since a solution of (A.4) on $\mathfrak{a} + itU_+$ is determined by the value at any one point, we conclude $\tilde{\Phi}_1 = \tilde{\Phi}_2$. \square

Lemma A.6. *Suppose $\alpha \in R_1^+$ and $H_1 \in \mathfrak{a} + iU$ satisfy*

$$\alpha(H_1) = 0, \quad \beta(H_1) \neq 0 \text{ for any } \beta \in R_1^+ \setminus \{\alpha\}.$$

Then there exists an open neighborhood Ω of H_1 such that $\tilde{\Phi}$ extends to a single-valued holomorphic function on $(\mathfrak{a} + iU)_{\text{reg}} \cup \Omega$.

Proof. We may assume $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ and $\alpha = w\alpha_1$ for some $w \in W$. Let $\{\xi_1, \dots, \xi_\ell\} \subset \mathfrak{a}$ be the dual basis of $\{w\alpha_1, \dots, w\alpha_\ell\} \subset \mathfrak{a}^*$. Take $\xi_0 \in \mathfrak{a}_+$ and put $H_0 = w\xi_0 + ws_{\alpha_1}\xi_0$. Then in the same way as the proof of the previous lemma one can prove for a sufficiently small $\epsilon > 0$

$$\{H_0 + z\xi_1; z \in \mathbb{C} \text{ with } 0 < |z| < \epsilon\} \subset (\mathfrak{a} + iU)_{\text{reg}},$$

the function $\Phi(H_0 + x\xi_1)$ of $x \in (-\epsilon, \epsilon)$ extends to a holomorphic function $\tilde{\Psi}(z)$ on $D := \{z \in \mathbb{C}; |z| < \epsilon\}$, and $\tilde{\Phi}(H_0 + z\xi_1) = \tilde{\Psi}(z)$ for any $z \in D \setminus \{0\}$.

Now put

$$Z = \{H \in \mathfrak{a} + iU; \alpha(H) = 0, \beta(H) \neq 0 \text{ for any } \beta \in R_1^+ \setminus \{\alpha\}\}.$$

Then Z is a complex submanifold of $\mathfrak{a} + iU$ containing H_0 and H_1 . Since Z is pathwise connected, by shrinking D if necessary, we can find a connected open subset $\omega \subset Z$ containing H_0 and H_1 so that

$$\Omega := \{H + z\xi_1; z \in D, H \in \omega\} \subset \{H \in \mathfrak{a} + iU; \beta(H) \neq 0 \text{ for any } \beta \in R_1^+ \setminus \{\alpha\}\}.$$

Then

$$\Omega \cap (\mathfrak{a} + iU)_{\text{reg}} = \{H + z\xi_1; z \in D \setminus \{0\}, H \in \omega\}.$$

Observe from (A.2) that A_j in (A.4) for the current ξ_j ($j = 2, \dots, \ell$) is holomorphic on Ω . Now for each fixed $z \in D$ let us consider the system

$$(A.5) \quad \partial(\xi_j)\Psi(z, H) = A_j(H + z\xi_1)\Psi(z, H) \quad (j = 2, \dots, \ell)$$

for a $\mathbb{C}W$ -valued holomorphic function $\Psi(z, \cdot)$ on ω with the initial condition

$$(A.6) \quad \Psi(z, H_0) = \tilde{\Psi}(z).$$

It is clear that $\Psi(z, H) := \tilde{\Phi}(H + z\xi_1)$ is the solution when $z \neq 0$. This $\Psi(z, H)$ is holomorphic on $(D \setminus \{0\}) \times \omega$ as a function in z and H . Using (A.5) and (A.6) we can extend $\Psi(z, H)$ to a holomorphic function on $D \times \omega$ (this is possible without the integrability of the system). Thus $\tilde{\Phi}$ extends to a holomorphic function on $(\mathfrak{a} + iU)_{\text{reg}} \cup \Omega$ by letting

$$\tilde{\Phi}(H + z\xi_1) = \Psi(z, H) \quad \text{for } (z, H) \in D \times \omega. \quad \square$$

Now put

$$X = \{H \in \mathfrak{a}_{\mathbb{C}}; \alpha(H) = \beta(H) = 0 \text{ for some two distinct } \alpha, \beta \in R_1^+\}.$$

By Lemma A.6, $\tilde{\Phi}$ extends to a holomorphic function on $(\mathfrak{a} + iU) \setminus X$. But since X is a finite union of linear subspaces of $\mathfrak{a}_{\mathbb{C}}$ with codimension ≥ 2 , $\tilde{\Phi}$ still extends to a holomorphic function on the whole $\mathfrak{a} + iU$. Hence by Lemma A.5 we have $\Phi = \tilde{\Phi}|_{\mathfrak{a}}$ and the analyticity of Φ on \mathfrak{a} . This completes the proof of Theorem A.1.

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